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FINITE ENERGY SURFACES AND THE CHORD PROBLEM

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CONTENTS

1. Introduction. A *contact form* on an odd-dimensional manifold M of dimension $2n+1$ is a 1-form λ such that the $(2n+1)$-form Ω, given by

$$\Omega = \lambda \wedge (d\lambda)^n,$$

defines a volume form on M. We observe that any manifold admitting a contact form is necessarily orientable and that a contact form defines a natural orientation.

Assume now that (M, λ) is a manifold together with a given contact form. First of all, we note that λ defines a $2n$-dimensional vector bundle over M. Indeed, consider $\xi \to M$, where ξ is given by

$$\xi_m = \ker(\lambda_m).$$

The linear functional $\lambda_m : T_mM \to \mathbf{R}$ is nonzero since $\lambda \wedge (d\lambda)^n$ defines a volume form. Hence we obtain a vector bundle. Moreover, by the properties of λ, we see that $\omega := d\lambda \,|\, (\xi \oplus \xi)$ is nondegenerate on each fibre. Clearly, $\omega : \xi_m \oplus \xi_m \to \mathbf{R}$ is also skew-symmetric and bilinear; hence it is a symplectic form on ξ_m. Therefore, (ξ, ω) is a symplectic vector bundle.

Since the dimension of M is odd, $d\lambda$ is degenerate on each fibre T_mM of the tangent bundle TM. But it is as good as it can be, since λ is a contact form. Therefore, we obtain a line bundle ℓ over M via the definition

$$\ell_m = \{p \in T_mM \,|\, d\lambda_m(p, q) = 0 \text{ for all } q \in \xi_m\}.$$

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We observe that this line bundle has a natural section X_λ defined by the set of equations

$$i_{X_\lambda} d\lambda = 0, \qquad i_{X_\lambda}\lambda = 1.$$

Summing up, a contact form λ on an odd-dimensional manifold M of dimension $2n+1$ defines a natural splitting of the tangent bundle TM of M into a line bundle $\ell \to M$ with a preferred section X_λ and a symplectic vector bundle (ξ, ω):

$$TM = (\ell, X_\lambda) \oplus (\xi, \omega).$$

In the following, we always denote the data associated to a pair (M, λ) by ξ, ω, ℓ, and X_λ. The vector bundle $\xi \to M$ without its symplectic structure is called a *contact structure* on M. The vector field X_λ is called the *Reeb vector field* associated to λ. We observe that, given the contact structure ξ, we can rediscover the conformal class of the symplectic structure. Namely, take any nowhere-vanishing 1-form τ with $\ker(\tau) = \xi$. Then

$$\lambda = f \cdot \tau$$

for some nonvanishing smooth function $f : M \to \mathbf{R}$. We observe that

$$\omega = d\lambda \,|\, (\xi \oplus \xi) = f \cdot d\tau \,|\, (\xi \oplus \xi).$$

If M is a $(2n+1)$-dimensional contact manifold with contact form λ, then the *symplectisation* of M is the manifold $\mathbf{R} \times M$ with symplectic structure $\omega := d(e^t \lambda)$. One verifies easily that ω is indeed a symplectic structure (recall that $d\lambda|_{\ker\lambda \oplus \ker\lambda}$ is nondegenerate). Assume now that J is a *compatible complex structure* on the bundle $\ker \lambda$, that is:

- $J(p) : \ker\lambda(p) \to \ker\lambda(p)$ is a linear map satisfying $J(p)^2 = -\mathrm{Id}$ for each $p \in M$;
- J depends smoothly on p;
- J satisfies the compatibility condition that $d\lambda \circ (\mathrm{Id} \times J)$ is a bundle metric on $\ker \lambda$.

(Such J exist; see [10].) We obtain in a natural way the following *almost complex structure on the symplectisation* $\mathbf{R} \times M$:

$$\tilde{J}(a, u)(h, k) := (-\lambda(u)k, J(u)\pi_\lambda k + h \cdot X_\lambda(u)), \tag{1}$$

where $(a, u) \in \mathbf{R} \times M$, $(h, k) \in \mathbf{R} \times T_u M$, and

$$\pi_\lambda = \pi_\lambda(u) : T_u M \to \ker \lambda(u)$$

$$\zeta \mapsto \zeta - (\lambda(u)\zeta) \cdot X_\lambda(u)$$

is the projection onto the contact structure along the Reeb vector field. We remark that $g_{\tilde{J}} := \omega \circ (\mathrm{Id} \times \tilde{J})$ is a Riemannian metric on $\mathbf{R} \times M$ (such a $\tilde{J}$ is called ω-compatible). If S is a Riemann surface with complex structure i, then we define a map

$$\tilde{u} = (a, u) : S \to \mathbf{R} \times M$$

to be a *pseudoholomorphic curve* if

$$T\tilde{u} \circ i = \tilde{J}(\tilde{u}) \circ T\tilde{u}.$$

If (s, t) are conformal coordinates on S, then this becomes

$$\partial_s \tilde{u} + \tilde{J}(\tilde{u}) \partial_t \tilde{u} = 0. \tag{2}$$

Using the expression (1) for $\tilde{J}$ and writing $\tilde{u} = (a, u)$, we obtain the following system of equations which is equivalent to (2):

$$\begin{aligned} \pi_\lambda \partial_s u + J(u) \pi_\lambda \partial_t u &= 0, \\ \partial_s a - \lambda(u) \partial_t u &= 0, \\ \partial_t a + \lambda(u) \partial_s u &= 0. \end{aligned} \tag{3}$$

Pseudoholomorphic curves (with compact Riemannian surface S) were introduced into symplectic geometry by M. Gromov in 1985 (see [4]) and became an important tool there. They were used in contact geometry by H. Hofer in 1993 (with $S = \mathbf{C}$) to prove existence of contractible periodic orbits of the Reeb vector field (see [6] or [1]).

We are interested in a different class of orbits of the Reeb vector field, the so-called *characteristic chords*. The terminology is due to V. I. Arnold [2] who raised a conjecture about the existence question. Let us give the definition: If (M, λ) is a contact manifold of dimension $2n + 1$, then a *Legendrian submanifold* is a submanifold $\mathscr{L}$ of M, which is n-dimensional and everywhere tangent to the contact structure $\ker \lambda$; that is, for each $p \in \mathscr{L}$, we have $\lambda(p)|_{T_p \mathscr{L}} \equiv 0$. Then a *characteristic chord* for $(\lambda, \mathscr{L})$ is a smooth path

$$x : [0, T] \to M, \ T > 0$$

with

- $\dot{x}(t) = X_\lambda(x(t)) \, \forall t \in (0, T)$,
- $x(0), x(T) \in \mathscr{L}$.

Arnold raised the following conjecture.

CONJECTURE (see [2]). *Let λ_0 be the standard tight contact form*

$$\lambda_0 = \frac{1}{2}(x_1\,dy_1 - y_1\,dx_1 + x_2\,dy_2 - y_2\,dx_2)$$

on the 3-sphere

$$S^3 = \{(x_1, y_1, x_2, y_2) \in \mathbf{R}^4 \mid x_1^2 + y_1^2 + x_2^2 + y_2^2 = 1\}.$$

If $f : S^3 \to (0, \infty)$ is a smooth function and $\mathscr{L}$ is a Legendrian knot in S^3, then there is a characteristic chord for $(f\lambda_0, \mathscr{L})$.

There is almost nothing known about this problem. Arnold only mentioned the case where $f \equiv 1$. Here are some simple examples.

Example 1. $\mathbf{R}^3$ with the standard tight contact form. We consider the contact form $\lambda = dz + x\,dy$ on $\mathbf{R}^3$. The Reeb vector field is just the constant vector field $\partial/\partial z$. If a closed curve $(x(t), y(t), z(t)), 0 \leqslant t \leqslant 1$, is Legendrian, then we must have

$$\dot{z}(t) = -\dot{y}(t)x(t)$$

and

$$\begin{aligned} 0 &= z(1) - z(0) \\ &= -\int_0^1 \dot{y}(t)x(t)\,dt \\ &= \int_0^1 y(t)\dot{x}(t)\,dt. \end{aligned}$$

Consider now the projection of the above curve onto the xy-plane. The (oriented) area of the set enclosed by the projected curve equals

$$\int_0^1 (x(t)\dot{y}(t) - \dot{x}(t)y(t))\,dt = 0,$$

and hence the projected curve must have a self-intersection, which implies the existence of a characteristic chord.

Example 2. $\mathbf{R}^3$ with the (standard) overtwisted contact form. In cylindrical coordinates, we consider the contact form $\lambda = \cos r\,dz + r\sin r\,d\phi$. Then for each $k \in \mathbf{Z}$, the knot

$$(r, \phi, z)(t) := (\pi k, t, 0)$$

with $0 \leqslant t \leqslant 2\pi$ is Legendrian, but the Reeb vector field on the cylinder $\{(\pi k, t, z) \mid z \in \mathbf{R}, 0 \leqslant t \leqslant 2\pi\}$ is given by

$$X_\lambda(\pi k, t, z) = (-1)^k \frac{\partial}{\partial z};$$

hence there is no characteristic chord for these particular knots.

Remarks. (1) If $M = S^3 \subset \mathbf{C}^2$ is endowed with the standard tight contact form (i.e., $f \equiv 1$), then we consider the Hopf-fibration

$$h : S^3 \to \mathbf{C}P^1$$

$$(z_0, z_1) \mapsto [z_0, z_1],$$

where $[z_0, z_1] := \{(z_0', z_1') \in S^3 \mid \exists \theta \in S^1 : z_0' = \theta z_0, z_1' = \theta z_1\}$. The fibres are exactly the orbits of the Reeb vector field. Let $\mathscr{L} \subset S^3$ be a Legendrian knot. Then $h(\mathscr{L})$ encloses a set whose area is an integer multiple of 4π. This means that $h(\mathscr{L})$ has a self-intersection, so there must be a characteristic chord for $\mathscr{L}$.

(2) Even if f is C^∞ near to 1, it is not at all clear that characteristic chords exist, since the dynamics of the Reeb vector field can change completely.

(3) If M is a strictly convex hypersurface in $\mathbf{R}^4$ with contact form $\lambda_0|M$, then there is an open-book decomposition with binding orbit P (see [9]). If $\mathscr{L}$ is a Legendrian knot that is not linked with P, then there is also a characteristic chord for $\mathscr{L}$. The proof is similar to the ones of examples (1) or (2), except that we use the Reeb flow to project the knot onto a fixed leaf of the open-book decomposition. The projected curve must also have a self-intersection.

(4) If M and P are the same as above, but P and $\mathscr{L}$ are linked, the answer to whether there is a characteristic chord is not known. We can only give an affirmative answer in the special case where M is an "irrational ellipsoid"; that is, where

$$M = \left\{ (z_1, z_2) \in \mathbf{C}^2 \mid \frac{|z_1|^2}{r_1^2} + \frac{|z_2|^2}{r_2^2} = 1 \right\}$$

with r_2^2/r_1^2 irrational and with contact form λ_0, but the proof is nontrivial (one proves the existence of a nonconstant finite energy half-plane).

If $H^+ := \{s + it \in \mathbf{C} \mid t \geqslant 0\}$ is the closed upper half of the complex plane, (M, λ) is a closed contact manifold, and $\mathscr{L} \subset M$ is a Legendrian submanifold, then we define a *finite energy half-plane* to be a map

$$\tilde{u} = (a, u) : H^+ \to \mathbf{R} \times M$$

that satisfies the following conditions:

(i) $\partial_s \tilde{u} + \tilde{J}(\tilde{u})\partial_t \tilde{u} = 0$ on $\mathring{H}^+$;
(ii) $\tilde{u}(\partial H^+) \subset \mathbf{R} \times \mathscr{L}$;
(iii) $u(H^+)$ is contained in a compact region $K \subset M$;
(iv) $\tilde{u}$ has finite energy

$$E(\tilde{u}) := \sup_{\phi \in \Sigma} \int_{H^+} \tilde{u}^* \, d(\phi\lambda) < +\infty$$

(where $\Sigma := \{\phi \in C^\infty(\mathbf{R}, [0,1]) \mid \phi' \geqslant 0\}$).

We prove in Section 2 that the existence of a nonconstant finite energy half-plane implies the existence of a characteristic chord for $(\lambda, \mathscr{L})$. More precisely, we have the following theorem.

THEOREM 1.1. *Let $\tilde{u}$ be a finite energy half-plane that is, in addition, nonconstant. Then $T := \int_{H^+} u^* d\lambda > 0$, and any sequence of positive real numbers tending to $+\infty$ has a subsequence $R_k \to +\infty$, so that the maps*

$$[0, T] \to M$$

$$t \mapsto u(R_k \, e^{\pi i (t/T)})$$

converge in C^∞ to some orbit x of the Reeb vector field X_λ with $x(0), x(T) \in \mathscr{L}$.

In the following, we only discuss the case $n = 1$; that is, (M, λ) is a 3-dimensional manifold, and $\mathscr{L}$ is a Legendrian knot. If x is a characteristic chord for $(\lambda, \mathscr{L})$, then the pair $(x, \mathscr{L})$ is called *nondegenerate* if

$$\ker \lambda(x(T)) = T\varphi_T(x(0))\, T_{x(0)}\mathscr{L} \oplus T_{x(T)}\mathscr{L},$$

where φ denotes the flow of the Reeb vector field X_λ. In Section 3 we prove the following refinement of Theorem 1.1 under the additional assumption that x is nondegenerate.

THEOREM 1.2. *Let $\tilde{u}$ be a nonconstant finite energy half-plane. If $R_k \to \infty$ is a sequence of positive numbers so that $u(R_k e^{\pi i (t/T)})$ converges to a nondegenerate characteristic chord x, then, in fact, we have*

$$\lim_{R \to \infty} u(R e^{\pi i (t/T)}) = x(t)$$

with convergence in $C^\infty([0, T])$.

In order to study the existence question for characteristic chords, it is important to understand the behaviour of finite energy half-planes near infinity. Assume that $\tilde{u}$ is a finite energy half-plane that satisfies the assumptions of

Theorem 1.2. Let $\phi : \mathbf{R} \times [0, T] \to H^+\backslash\{0\}$ be the biholomorphic map $(s, t) \mapsto e^{(\pi/T)(s+it)}$. Considering the *finite energy strip* $\tilde{v} := \tilde{u} \circ \phi$ instead of $\tilde{u}$, we have

- $\partial_s\tilde{v} + \tilde{J}(\tilde{v})\partial_t\tilde{v} = 0$,
- $v(\mathbf{R} \times \{0, T\}) \subset \mathscr{L}$, where $v := u \circ \phi$,
- $E(\tilde{v}) < \infty$,
- $v(\mathbf{R} \times [0, T])$ is contained in a compact region in M,
- $\lim_{s\to\infty} v(s, t) = x(t)$.

Hence, by Theorem 1.2, we can study $v : [s_0, \infty) \times [0, T] \to M$ in a neighbourhood of $x([0, T])$ if s_0 is sufficiently large. The other important ingredient for further study of $(s, t) \mapsto \tilde{v}(s, t)$ for large s is a local coordinate description of a neighbourhood of $x([0, T])$ (Lemma 3.2). Then we may assume that

- v has image in $\mathbf{R}^3$,
- the Reeb vector field is parallel to the z-axis,
- the characteristic chord is given by $t \mapsto (0, 0, t)$,
- the boundary condition is $v(s, 0) \in \mathbf{R} \cdot (1, 0, 0)$ and $v(s, T) \in \mathbf{R} \cdot (0, 1, 0) + (0, 0, T)$.

(We are only interested in characteristic chords that satisfy $x(0) \neq x(T)$.) Writing $\tilde{v}(s, t) = (b, x, y, z)(s, t) \in \mathbf{R} \times \mathbf{R}^3$ for the components of $\tilde{v}$, we prove the following result, which states that the convergence of $v(s, t)$ to $x(t)$ is of exponential nature.

THEOREM 1.3. *Let* $\tilde{v}: [s_0, \infty) \times [0, T] \to \mathbf{R} \times \mathbf{R}^3$ *be a finite energy strip as explained above. Then there are constants* $b_0 \in \mathbf{R}$, $r > 0$, $s_0 \in \mathbf{R}$ *so that, for each multi-index* $\alpha = (\alpha_1, \alpha_2) \in \mathbf{N}^2$, $0 < \rho < \min\{r/2, 1/T\}$, *and* $s \geqslant s_0$ *we have,*

$$\sup_{0\leqslant t\leqslant T} |\partial^\alpha x(s, t)| \leqslant c_\alpha^1 e^{-rs},$$

$$\sup_{0\leqslant t\leqslant T} |\partial^\alpha y(s, t)| \leqslant c_\alpha^1 e^{-rs},$$

$$\sup_{0\leqslant t\leqslant T} |\partial^\alpha (z(s, t) - t)| \leqslant c_\alpha^2 e^{-\rho s},$$

$$\sup_{0\leqslant t\leqslant T} |\partial^\alpha (b(s, t) - b_0 - s)| \leqslant c_\alpha^3 e^{-\rho s},$$

where $\partial^\alpha = (\partial/\partial s)^{\alpha_1}(\partial/\partial t)^{\alpha_2}$ *and* $c_\alpha^1, c_\alpha^2, c_\alpha^3 > 0$ *are suitable constants (we adopt the convention that zero is contained in* $\mathbf{N}$*).*

If we denote by $\zeta = (x, y)$ the components of v that are transversal to the characteristic chord, we even get an asymptotic formula for ζ. Before we can state the result, we define an unbounded linear operator A_∞ with domain of definition $W_\Gamma^{1,2}([0, T], \mathbf{R}^2)$, which we define to be the set of paths $\gamma : [0, T] \to \mathbf{R}^2$ of class $W^{1,2}$ satisfying the boundary conditions $\gamma(0) \in \mathbf{R} \cdot (1, 0)$ and $\gamma(T) \in$

$\mathbf{R} \cdot (0,1)$. Then we define

$$A_\infty : L^2([0,T],\mathbf{R}^2) \supset W_\Gamma^{1,2}([0,T],\mathbf{R}^2) \to L^2([0,T],\mathbf{R}^2)$$

by

$$(A_\infty \cdot \gamma)(t) := -M_\infty(t)\dot{\gamma}(t),$$

where we abbreviate $M_\infty(t) := M(0,0,t)$.

THEOREM 1.4. *If ζ does not vanish identically, we have the asymptotic formula*

$$\zeta(s,t) = \exp\left(\int_{s_0}^{s} \alpha(\tau)\, d\tau\right)[e(t) + r(s,t)],$$

where

- $e \in W_\Gamma^{1,2}([0,T],\mathbf{R}^2)$ *is an eigenvector of A_∞ corresponding to some eigenvalue $\lambda < 0$ (here $L^2([0,T],\mathbf{R}^2)$ with the equivalent inner product $(\,.\,,\,.\,) = \int_0^T \langle\,.\,, -J_0 M_\infty(t).\rangle\, dt$) and A_∞ is selfadjoint;*
- $\alpha : [s_0,\infty) \to \mathbf{R}$ *is a smooth function satisfying $\alpha(s) \to \lambda$ as $s \to \infty$;*
- $r : [s_0,\infty) \times [0,T] \to \mathbf{R}^2$ *is a smooth map with*

$$|\partial^\alpha r(s,t)| \to 0,$$

as $s \to \infty$ uniformly in t, and where $\alpha \in \mathbf{N}^2$ is some multi-index (recall that, by convention, $0 \in \mathbf{N}$).

It is important for the proofs of both Theorems 1.3 and 1.4 that ζ solves a boundary value problem of the following type:

$$\begin{gathered} \partial_s\zeta(s,t) + M(\zeta(s,t),z(s,t))\partial_t\zeta(s,t) = 0, \\ \zeta(s,0) \in \mathbf{R}\cdot(1,0), \\ \zeta(s,T) \in \mathbf{R}\cdot(0,1), \end{gathered} \tag{4}$$

where M is a matrix-valued function defined near $\{(0,0)\} \times [0,T] \subset \mathbf{R}^3$ with

- $M^2 = -\mathrm{Id}$,
- $M^T J_0 M = J_0$ with $J_0 = \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}$,
- $-J_0 M > 0$.

Defining the differential operator $A(s)$ acting on paths $\gamma \in W_\Gamma^{1,2}([0,T],\mathbf{R}^2)$ by

$$(A(s)\gamma)(t) := -M(\zeta(s,t),z(s,t))\dot{\gamma}(t),$$

we can write equation (4) as follows ($\zeta(s) := \zeta(s, .)$):

$$\partial_s \zeta(s) = A(s) \cdot \zeta(s).$$

One can find inner products $(. , .)_s$ on $L^2([0, T], \mathbf{R}^2)$, which are equivalent to the ordinary L^2-product, so that $A(s)$ becomes a selfadjoint operator on

$$(L^2([0, T], \mathbf{R}^2), (. , .)_s).$$

This is essential for the proofs of both Theorems 1.3 and 1.4. In [7] and [8], where similar results are derived for finite energy planes, the authors can carry out a change of coordinates so that $M(s, t)$, $A(s)$, and $(. , .)_s$ are transformed into s-independent quantities. In contrast to [7] and [8], we are dealing with a boundary value problem (4), and any change of coordinates removing the s-dependence in $M(s, t)$, $A(s)$, or the L^2-product will make the boundary condition nonlinear.

So we have to work with an L^2 inner product that depends on s and a family of operators $(A(s))_{s \geqslant s_0}$, instead of just one operator. This is the main reason that Theorems 1.3 and 1.4 have to be proved differently than their counterparts in [7] and [8].

2. Finite energy half-planes and orbits of the Reeb vector field. We describe now the relationship between pseudoholomorphic half-planes of finite energy and characteristic chords. The main theorem of this chapter is as follows.

THEOREM 2.1. *Let $\tilde{u}$ be a finite energy half-plane as above, which is, in addition, nonconstant. Then $T := \int_{H^+} u^* \, d\lambda > 0$, and any sequence of positive real numbers tending to $+\infty$ has a subsequence $R_k \to +\infty$, so that the maps*

$$[0, T] \to M$$

$$t \mapsto u(R_k \, e^{\pi i (t/T)})$$

converge in C^∞ to some orbit x of the Reeb vectorfield X_λ with $x(0), x(T) \in \mathscr{L}$.

The proof requires some preparation. The following proposition states that finite energy half-planes with "trivial $d\lambda$-energy" must be constant. The boundary condition plays an important role in the proof.

PROPOSITION 2.2. *Let $\tilde{u}$ be a finite energy half-plane with*

$$\int_{H^+} u^* \, d\lambda = 0.$$

Then $\tilde{u}$ must be constant.

Proof. With $z = s + it$, we compute

$$0 = \int_{H^+} u^* \, d\lambda = \frac{1}{2} \int_{H^+} (|\pi_\lambda \partial_s u|_J^2 + |\pi_\lambda \partial_t u|_J^2) \, ds \wedge dt.$$

Hence $\pi_\lambda \circ Tu(z) : H^+ \to \ker \lambda(u(z))$ is the zero map for all $z \in H^+$. Moreover,

$$\begin{aligned} 0 &= \frac{1}{2}(|\pi_\lambda \partial_s u|_J^2 + |\pi_\lambda \partial_t u|_J^2) \, ds \wedge dt \\ &= u^* \, d\lambda \\ &= -d(da \circ i) \\ &= \Delta a \cdot ds \wedge dt; \end{aligned}$$

hence a is a harmonic map. Define now

$$f : H^+ \to \mathbf{R}$$

$$f(s,t) := \int_0^t \partial_s a(s,\tau) \, d\tau,$$

and note that $\partial_t a(s,0) = -\lambda(u(s,0))\partial_s u(s,0) = 0$ because of the boundary condition. Then

$$\Phi^+ := a + if : H^+ \to \mathbf{C}.$$

is a holomorphic function with $\Phi^+(\partial H^+) \subset \mathbf{R}$. We define the holomorphic function

$$\Phi : \mathbf{C} \to \mathbf{C}$$

$$\Phi(z) := \begin{cases} \Phi^+(z) & \text{if } z \in H^+, \\ \overline{\Phi^+(\bar{z})} & \text{if } z \in \mathbf{C} \backslash H^+. \end{cases}$$

Now, with $\phi \in \Sigma$ and $\tau_\phi := d(\phi(s) \, dt) = \phi'(s) \, ds \wedge dt$, compute

$$\begin{aligned} \int_{H^+} (\Phi^+)^* \tau_\phi &= \int_{H^+} \phi'(a)(\partial_s a^2 + \partial_t a^2) \, ds \wedge dt \\ &= \int_{H^+} \tilde{u}^* \, d(\phi\lambda) \\ &< +\infty. \end{aligned}$$

We distinguish the following three cases:

(1) Φ^+ is constant.

(2) $|\nabla\Phi^+|$ is bounded, but Φ^+ is not constant.

(3) $|\nabla\Phi^+|$ is unbounded.

Case 1. If Φ^+ is constant, then a is constant, too. Since

$$\lambda(u)\partial_s u = -\partial_t a = 0$$

and

$$\lambda(u)\partial_t u = \partial_s a = 0,$$

the map $Tu(z)$ has an image in the kernel of $\lambda(u(z))$. However, we saw before that $\pi_\lambda \circ Tu(z)$ is the zero map, so $Tu(z)$ is always zero and u is constant.

We show now that the other two cases cannot occur.

Case 2. If $|\nabla\Phi^+|$ is bounded, then $|\nabla\Phi|$ is also bounded. By Liouville's theorem, the functions $\partial_s\Phi, \partial_t\Phi : \mathbf{C} \to \mathbf{C}$ must be constant; hence, Φ must be an affine function

$$\Phi(z) = \alpha z + \beta$$

with $\beta \in \mathbf{C}$ and $\alpha \in \mathbf{C}\backslash\{0\}$. Since $\Phi(\partial H^+) = \Phi^+(\partial H^+) \subset \mathbf{R}$, the numbers α and β must be real. We obtain

$$\begin{aligned}\int_{H^+} (\Phi^+)^*\tau_\phi &= \int_{H^+} \phi'(a)(\partial_s a^2 + \partial_t a^2)\, ds \wedge dt \\ &= \alpha^2 \int_{H^+} \phi'(a)\, ds \wedge dt \\ &= |\alpha| \int_{H^+} \phi'(\sigma)\, d\sigma \wedge dt \\ &= +\infty \quad \text{for any nonconstant } \phi \in \Sigma\end{aligned}$$

in contradiction to $\int_{H^+} \tilde{u}(d(\phi\lambda)) < +\infty$.

Case 3. If $|\nabla\Phi^+|$ is unbounded, we can pick sequences $(z'_k) \subset H^+$ and $\varepsilon'_k \searrow 0$ so that

$$R'_k := |\nabla\Phi(z'_k)| \to +\infty$$

and

$$R'_k\varepsilon'_k \to +\infty.$$

Applying a well-known lemma of Hofer's (see [1] and [5]), we find sequences $\varepsilon_k \searrow 0$ and $(z_k) \subset H^+$ with

- $\varepsilon_k R_k := \varepsilon_k |\nabla\Phi^+(z_k)| \geqslant \varepsilon_k' R_k'$,
- $|z_k' - z_k| \leqslant \varepsilon_k'$,
- $|\nabla\Phi^+(y)| \leqslant 2\,|\nabla\Phi^+(z_k)|\,\forall y$ with $|y - z_k| \leqslant \varepsilon_k$.

We have to consider the following cases, where $z_k = s_k + it_k$:

- $t_k R_k \to +\infty$ (without loss of generality, assume $t_k R_k \nearrow +\infty$),
- $t_k R_k \to l \in [0, +\infty)$.

Let us begin with the first case. Define the holomorphic maps

$$\Phi_k(z) := \Phi^+\left(z_k + \frac{z}{R_k}\right) - \Phi^+(z_k),$$

which are defined on

$$\Omega_k := \{(s,t) \in \mathbf{C} \mid t \geqslant -t_k R_k\}.$$

We compute

$$\nabla\Phi_k(z) = \frac{1}{|\nabla\Phi^+(z_k)|}\left|\nabla\Phi^+\left(z_k + \frac{z}{R_k}\right)\right|.$$

Hence

$$|\nabla\Phi_k(0)| = 1,$$

$$|\nabla\Phi_k(z)| \leqslant 2 \quad \text{for } z \in B_{\varepsilon_k R_k}(0),$$

$$\Phi_k(0) = 0.$$

Let K be a compact subset of the complex plane. Choose k_0 so large so that for all $k \geqslant k_0$ we have $K \subset \Omega_k$ and $K \subset B_{\varepsilon_k R_k}(0)$. Then $(\Phi_k)_{k \geqslant k_0}$ is a sequence of nonconstant holomorphic functions on K, which is uniformly bounded in C^1. Using the Cauchy integral formula, we obtain uniform C^∞-bounds on K. By the Ascoli-Arzela theorem, there exists a subsequence $(\Phi_{k'}) \subset (\Phi_k)$ that converges in $C^\infty(K)$. Iterating this process by taking larger K and extracting further subsequences from $(\Phi_{k'})$, we get, by choosing a diagonal sequence, some subsequence of (Φ_k) that converges in C^∞_{loc} to a holomorphic map

$$\Psi : \mathbf{C} \to \mathbf{C}$$

satisfying $|\nabla\Psi(0)| = 1$ and $|\nabla\Psi(z)| \leqslant 2$. By Liouville's theorem, Ψ must be an affine (nonconstant) function. Defining $\phi_k(s) := \phi(s - \mathrm{Re}(\Phi^+(z_k))) \in \Sigma$, we have

$$\begin{aligned}
+\infty &> E(\tilde{u}) \\
&\geqslant \int_{H^+} \tilde{u}^* \, d(\phi_k \lambda) \\
&= \int_{H^+} (\Phi^+)^* \tau_{\phi_k} \\
&= \int_{\Omega_k} \Phi_k^* \tau_\phi.
\end{aligned}$$

On every compact $K \subset \mathbf{C}$, we have

$$\int_K \Phi_k^* \tau_\phi \to \int_K \Psi^* \tau_\phi$$

for $k \to +\infty$. It follows that, for any nonconstant $\phi \in \Sigma$,

$$\begin{aligned}
+\infty &> E(\tilde{u}) \\
&\geqslant \int_{\mathbf{C}} \Psi^* \tau_\phi \\
&= \int_{\mathbf{C}} \tau_\phi \\
&= \int_{\mathbf{C}} \phi'(s) \, ds \wedge dt \\
&= +\infty.
\end{aligned}$$

This contradiction shows that the first case cannot occur. So let us consider the second case.

Here we define the following holomorphic maps, where $z = s + it$ and $z_k = s_k + it_k$:

$$\begin{aligned}
\Phi_k(z) &:= \Phi^+\left(\mathrm{Re}\, z_k + \frac{z}{R_k}\right) - \Phi^+(\mathrm{Re}\, z_k) \\
&= \Phi^+\left(s_k + \frac{s}{R_k}, \frac{t}{R_k}\right) - \Phi^+(s_k, 0).
\end{aligned}$$

They are defined on the upper half-plane H^+ and map the boundary ∂H^+ into the real numbers. We calculate

$$|\nabla\Phi_k(s,t)| = \frac{1}{|\nabla\Phi^+(s_k,t_k)|}\left|\nabla\Phi^+\left(s_k+\frac{s}{R_k},\frac{t}{R_k}\right)\right|$$

so that

$$|\nabla\Phi_k(0,R_k t_k)| = 1$$

and

$$|\nabla\Phi_k(s,t)| \leqslant 2$$

for all $(s,t)\in B_{\varepsilon_k R_k}(0,R_k t_k)\cap H^+$. For any compact subset K of H^+, we can find some number k_0 so that for all $k \geqslant k_0$ we have $K \subset B_{\varepsilon_k R_k}(0,R_k t_k)$. Since $\Phi(0)=0$, we obtain a uniform C^1-bound for the sequence $(\Phi_k)_{k\geqslant k_0}$. Reasoning as before, we obtain C^∞_{loc} convergence of some subsequence of (Φ_k) to a nonconstant holomorphic map

$$\Psi : H^+ \to \mathbf{C},$$

which satisfies $|\nabla\Psi(z)| \leqslant 2$ and $\Psi(\partial H^+)\subset \mathbf{R}$. Using the Schwarz reflection principle, we can extend Ψ to an entire holomorphic function that must be affine by Liouville's theorem. $H := \Psi(H^+)$ is then again a half-plane in $\mathbf{C}$, and we compute, as before,

$$\begin{aligned}+\infty &> E(\tilde{u})\\ &\geqslant \int_{H^+}(\Phi^+)^*\tau_{\phi_k}\\ &= \int_{H^+}(\Phi_k^+)^*\tau_\phi,\end{aligned}$$

where $\phi_k := \phi(\,.\,-\operatorname{Re}\Phi^+(s_k,0))\in\Sigma$. On every compact subset K of H^+, we have

$$\lim_{k\to+\infty}\int_K \Phi_k^*\tau_\phi = \int_K \Psi^*\tau_\phi.$$

For any nonconstant $\phi\in\Sigma$, we obtain the contradiction

$$
\begin{aligned}
E(\tilde{u}) &\geqslant \int_{H^+} \Psi^* \tau_\phi \\
&= \int_H \tau_\phi \\
&= \int_H \phi'(s)\, ds \wedge dt \\
&= +\infty,
\end{aligned}
$$

which finally proves the proposition. □

The next proposition shows that the gradient of a finite energy half-plane is bounded in C^0. For convenience, we transform a finite energy half-plane $\tilde{u}$ to the infinite strip $\mathbf{R} \times [0,1]$ by considering $\tilde{v} := \tilde{u} \circ \phi$ instead of $\tilde{u}$, where ϕ is the biholomorphic map

$$
\phi : \mathbf{R} \times [0,1] \to H^+ \backslash \{0\}
$$

$$
\phi(s,t) := e^{\pi(s+it)}.
$$

We call $\tilde{v}$ a *finite energy strip* since $E(\tilde{v}) = E(\tilde{u}) < \infty$.

PROPOSITION 2.3. *Let* $\tilde{v} = (b, v) : \mathbf{R} \times [0,1] \to \mathbf{R} \times M$ *be a solution of the boundary value problem*

$$
\partial_s \tilde{v} + \tilde{J}(\tilde{v}) \partial_t \tilde{v} = 0,
$$

$$
v(\mathbf{R} \times \{0,1\}) \subset \mathscr{L}.
$$

Assume, moreover, that $v(\mathbf{R} \times [0,1])$ *is contained in a compact region* $K \subset M$ *and*

$$
E(\tilde{v}) < +\infty.
$$

Then

$$
\sup_{(s,t) \in \mathbf{R} \times [0,1]} |\nabla \tilde{v}(s,t)| < +\infty.
$$

Proof. Assume there are $z_k' = (s_k', t_k') \in \mathbf{R} \times [0,1]$ with

$$
R_k' := |\nabla \tilde{v}(s_k', t_k')| \to +\infty.
$$

Then we must have $s_k \to +\infty$. Otherwise, if we had $s_k \leqslant c$ for a subsequence, then the sequence $\phi(z_k')$ would be contained in some half-ball $B_R^+(0)$, but $\sup_{B_R^+(0)}|\nabla\tilde{u}| \leqslant \text{const}$ and $|\nabla\tilde{v}(z)| = \pi e^{\pi s}|\nabla\tilde{u}(\phi(z))|$. This is a contradiction since this expression would be bounded.

We may assume without loss of generality that the sequence (t_k') converges to some $t_0 \in [0,1]$. Choose now a sequence (ε_k') of positive real numbers converging to zero so that still $R_k'\varepsilon_k' \to +\infty$. Hofer's lemma gives us now new sequences $\varepsilon_k \searrow 0$ and $(z_k) \subset \mathbf{R} \times [0,1]$ with

- $\varepsilon_k R_k := \varepsilon_k|\nabla\tilde{v}(z_k)| \geqslant \varepsilon_k' R_k'$,
- $|z_k - z_k'| \leqslant \varepsilon_k'$,
- $|\nabla\tilde{v}(y)| \leqslant 2\,|\nabla\tilde{v}(z_k)|\,\forall y$ with $|y - z_k| \leqslant \varepsilon_k$.

Define now

$$\begin{aligned}\tilde{v}_k(s,t) &:= (b_k(s,t), v_k(s,t)) \\ &:= \left(b\left(z_k + \frac{z}{R_k}\right) - b(z_k), v\left(z_k + \frac{z}{R_k}\right)\right),\end{aligned}$$

where $z = (s,t)$ is contained in

$$\Omega_k := \{(s,t) \in \mathbf{C} \mid -t_k R_k \leqslant t \leqslant R_k(1-t_k)\}.$$

We have

(1) $|\nabla\tilde{v}_k(0)| = 1$,
(2) $|\nabla\tilde{v}(z)| \leqslant 2\ \forall z \in B_{\varepsilon_k R_k}(0) \cap \Omega_k$,
(3) $b_k(0) = 0$,
(4) $v_k(\partial\Omega_k) \subset \mathscr{L}$.

Because of (2), (3), and the assumption $v(\mathbf{R} \times [0,1]) \subset K$, we have a C^0_{loc}-bound on all the maps $\tilde{v}_k$ uniform in k. Using the usual regularity estimates (see, for example, [1]), we get uniform C^∞_{loc}-bounds. This implies, by the Ascoli-Arzela theorem, that a subsequence of $\tilde{v}_k$ converges in C^∞_{loc} to some limit

$$\tilde{w} = (\beta, w) : \Omega \to \mathbf{R} \times M,$$

where $\Omega \subset \mathbf{C}$ depends on the following cases:

(1) We have $-t_k R_k \to l \in (-\infty, 0]$ (then necessarily $R_k(1-t_k) \to +\infty$), where we have $\Omega = H_{-l} := \{z \in \mathbf{C} \mid \text{Im}(z) \geqslant -l\}$ and $w(\partial H_{-l}) \subset \mathscr{L}$.

(2) We have $-t_k R_k \to -\infty$.

(a) For

$$R_k(1-t_k) \to m \in [0, +\infty),$$

we have $\Omega = H^m := \{z \in \mathbf{C} \mid \text{Im}(z) \leqslant m\}$ and $w(\partial H^m) \subset \mathscr{L}$;

(b) for

$$R_k(1-t_k) \to +\infty,$$

we get $\Omega = \mathbf{C}$.

In all these cases, we have

$$\partial_s \tilde{w} + \tilde{J}(\tilde{w})\partial_t \tilde{w} = 0,$$

$$|\nabla \tilde{w}(0)| = 1,$$

$$|\nabla \tilde{w}(z)| \leqslant 2.$$

We claim that

- $E(\tilde{w}) \leqslant E(\tilde{v})$,
- $\int_\Omega w^* \, d\lambda = 0.$

Take $\phi \in \Sigma$, and define $\phi_k \in \Sigma$ by

$$\phi_k(s) := \phi(s - b(z_k)).$$

Then

$$\begin{aligned} \int_{B_{R_k \varepsilon_k}(0) \cap \Omega_k} \tilde{v}_k^* \, d(\phi\lambda) &= \int_{B_{\varepsilon_k}(z_k) \cap (\mathbf{R} \times [0,1])} \tilde{v}^* \, d(\phi_k \lambda) \\ &\leqslant \int_{\mathbf{R} \times [0,1]} \tilde{v}^* \, d(\phi_k \lambda) \\ &\leqslant E(\tilde{v}). \end{aligned}$$

Now choose any compact subset K of Ω and find $k_0 \in \mathbf{N}$ so that, for all $k \geqslant k_0$:

$$K \subset B_{R_k \varepsilon_k}(0) \cap \Omega_k.$$

Then

$$\int_K \tilde{v}_k^* \, d(\phi\lambda) \leqslant E(\tilde{v}) \quad \forall k \geqslant k_0,$$

and therefore

$$\int_K \tilde{w}^* \, d(\phi\lambda) \leqslant E(\tilde{v}).$$

Since this holds for all compact subsets K of Ω, we obtain

$$\int_\Omega \tilde{w}^* \, d(\phi\lambda) \leqslant E(\tilde{v}),$$

and, finally, taking the supremum over all $\phi \in \Sigma$,

$$E(\tilde{w}) \leqslant E(\tilde{v}).$$

Now let K be any compact subset of Ω. Then, for k large enough, we have $K \subset B_{R_k\varepsilon_k}(0) \cap \Omega_k$ and

$$\begin{aligned}\int_K w^* \, d\lambda &\leqslant \left| \int_K w^* \, d\lambda - \int_K v_k^* \, d\lambda \right| + \int_{B_{R_k\varepsilon_k}(0)\cap\Omega_k} v_k^* \, d\lambda \\ &\leqslant \left| \int_K w^* \, d\lambda - \int_K v_k^* \, d\lambda \right| + \int_{B_{\varepsilon_k}(z_k)\cap(\mathbf{R}\times[0,1])} v^* \, d\lambda.\end{aligned}$$

The first term converges to zero for $k \to +\infty$, but the second one also does because of

$$\int_{\mathbf{R}\times[0,1]} v^* \, d\lambda = \int_{\mathbf{R}\times[0,1]} \tilde{v}^* \, d(\phi_0\lambda) \leqslant E(\tilde{v}) < +\infty,$$

where $\phi_0 \equiv 1 \in \Sigma$. This implies finally

$$\int_\Omega w^* \, d\lambda = 0,$$

because the integral vanishes over any compact subset of Ω. If Ω is a half-plane in $\mathbf{C}$, then Proposition 2.2 would imply that $\tilde{w}$ must be constant, which contradicts the fact that the gradient in zero does not vanish. Hence our assumption at the very beginning (that the gradient is unbounded) must be false.

Since Proposition 2.2 also holds for finite energy planes (see [1] or [6]; the proof is simpler than for half-planes), we also arrive at a contradiction in the case $\Omega = \mathbf{C}$, which finishes the proof of the proposition. □

Now we are ready to prove Theorem 2.1.

We assume that we have transformed our finite energy plane to a finite energy strip $\tilde{v} = (b, v)$. So take any sequence $s_k \to +\infty$ and define

$$\tilde{v}_k : \mathbf{R} \times [0,1] \to \mathbf{R} \times M$$

$$\begin{aligned}\tilde{v}_k(s,t) &:= (b(s+s_k,t) - b(s_k,0), v(s_k+s,t)) \\ &=: (b_k(s,t), v_k(s,t)).\end{aligned}$$

Then

$$b_k(0,0) = 0$$

and

$$v_k(\mathbf{R} \times \{0,1\}) \subset \mathscr{L}.$$

As before, we get C^∞_{loc}-bounds, and a subsequence of $(\tilde{v}_k)$ converges in C^∞_{loc} to some

$$\tilde{w} = (\beta, w) : \mathbf{R} \times [0,1] \to \mathbf{R} \times M,$$

satisfying

- $\partial_s \tilde{w} + \tilde{J}(\tilde{w}) \partial_t \tilde{w} = 0$,
- $w(\mathbf{R} \times \{0,1\}) \subset \mathscr{L}$,
- $E(\tilde{w}) < +\infty$,
- $\beta(0,0) = 0$,
- $\sup_{(s,t) \in \mathbf{R}\times[0,1]} |\nabla \tilde{w}(s,t)| < +\infty$.

Fix $s_0 \in \mathbf{R}$. If $-R < \min\{s_0 + s_k, 0\}$, then

$$\begin{aligned}\int_{[-R,s_0+s_k]\times[0,1]} v^*\,d\lambda &= -\int_{\{-R\}\times[0,1]} v^*\lambda + \int_{\{s_0+s_k\}\times[0,1]} v^*\lambda \\ &\quad + \int_{[-R,s_0+s_k]\times\{0\}} v^*\lambda - \int_{[-R,s_0+s_k]\times\{1\}} v^*\lambda \\ &= \int_{\{s_0+s_k\}\times[0,1]} v^*\lambda - \int_{\{-R\}\times[0,1]} v^*\lambda\end{aligned}$$

because of the Legendrian boundary condition. Moreover, the second term tends to zero as $R \to +\infty$, since $v(s,t)$ converges to a point as $s \to -\infty$. Hence

$$\begin{aligned}\int_{(-\infty,s_0+s_k]\times[0,1]} v^*\,d\lambda &= \int_{\{s_0+s_k\}\times[0,1]} v^*\lambda \\ &= \int_{\{s_0\}\times[0,1]} v_k^*\lambda\end{aligned}$$

and

$$\int_{\{s_0\}\times[0,1]} w^*\lambda = \int_{\mathbf{R}\times[0,1]} v^*\,d\lambda = \int_{H^+} u^*\,d\lambda =: T > 0.$$

For every $R > 0$ we have

$$\int_{[-R,R]\times[0,1]} v_k^*\,d\lambda = \int_{[-R+s_k,R+s_k]\times[0,1]} v^*\,d\lambda,$$

but this converges to zero as $k \to \infty$ because the integrand is nonnegative and

$$0 < \int_{\mathbf{R} \times [0,1]} v^* \, d\lambda < \infty.$$

Hence $\int_{[-R,R] \times [0,1]} w^* \, d\lambda = 0$ for every $R > 0$, and therefore

$$\int_{\mathbf{R} \times [0,1]} w^* \, d\lambda = 0.$$

As in the proof of Proposition 2.2, we see that $\pi_\lambda \circ Tw(z)$ is the zero map, and therefore $\Delta\beta = 0$. We note that, due to the boundary condition

$$w(\mathbf{R} \times \{0,1\}) \subset \mathscr{L},$$

$\partial_t \beta = -\lambda(w)\partial_s w$ must vanish identically on $\mathbf{R} \times \{0,1\}$. Our aim is to show that β must be an affine function depending on s only. Let us see first that this implies the existence of a characteristic chord.

Assume that

$$\beta(s,t) = as + b$$

with $a, b \in \mathbf{R}$. Then

$$\begin{aligned} \partial_s w &= \pi_\lambda \partial_s w + (\lambda(w)\partial_s w) \, X_\lambda(w) \\ &= -\partial_t \beta \cdot X_\lambda(w) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} \partial_t w &= \pi_\lambda \partial_t w + (\lambda(w)\partial_t w) \, X_\lambda(w) \\ &= \partial_s \beta \cdot X_\lambda(w) \\ &= a \, X_\lambda(w) \\ &\neq 0, \end{aligned}$$

since $\beta \equiv \text{const}$ would also imply that w is constant in contradiction to $\int_{\{s_0\} \times [0,1]} w^* \lambda > 0$. Hence

$$x(t) := w\left(s, \frac{t}{a}\right)$$

satisfies

$$\dot{x}(t) = \frac{1}{a}\partial_t w\left(s, \frac{t}{a}\right)$$
$$= X_\lambda(x(t)).$$

We compute

$$T = \int_{\{s_0\}\times[0,1]} w^*\lambda = \int_0^1 (\lambda(w)\partial_t w)\, dt = a,$$

and therefore

$$\tilde{w}(s,t) = (Ts + c, x(Tt)).$$

By construction,

$$v(s_k, t) \to x(Tt) \quad \text{in } C^\infty([0,1]),$$

which is equivalent to

$$u(e^{\pi(s_k+it)}) \to x(Tt)$$

and, replacing t by t/T,

$$u(e^{\pi(s_k+i(t/T))}) \to x(t) \quad \text{in} \quad C^\infty([0,T]),$$

where $x(0) = w(s,0) \in \mathscr{L}$ and $x(T) = w(s,1) \in \mathscr{L}$; hence x is a characteristic chord.

We are left with the proof that β is an affine function depending on s only. We put $\gamma := \partial_t\beta$ and recall that

- $\Delta\gamma = 0$,
- $C := \sup_{\mathbf{R}\times[0,1]}|\gamma| < \infty$,
- $\gamma(s,0) \equiv \gamma(s,1) \equiv 0$.

If we define

$$\delta(s,t) := \int_0^t \partial_s\gamma(s,\tau)\, d\tau - \int_0^s \partial_t\gamma(\sigma,0)\, d\sigma,$$

then $f := \gamma + i\delta$ is a holomorphic function with a bounded real part; $g(s,t) := e^{f(s,t)}$ is also holomorphic with

$$|g(s,t)| = |e^{\gamma(s,t)} \cdot e^{i\delta(s,t)}| \leqslant e^C$$

and

$$|g(s,0)| = |g(s,1)| = 1.$$

For each $\varepsilon > 0$, we define the holomorphic function h_ε on $\mathbf{R} \times [0,1]$ by

$$h_\varepsilon(z) := \frac{1}{1 - i\varepsilon z}.$$

Then

$$|h_\varepsilon(z)|^2 = \frac{1}{\varepsilon^2 s^2 + (1 + \varepsilon t)^2} \leqslant 1,$$

and therefore

$$|g(z)h_\varepsilon(z)| \leqslant 1$$

for $z \in \mathbf{R} \times \{0,1\}$. We also have, for $z \notin i\mathbf{R}$,

$$|h_\varepsilon(z)| \leqslant \frac{1}{\varepsilon\,|s|}$$

in view of $|1 - i\varepsilon z|^2 \geqslant \varepsilon^2 s^2$. Consider now the holomorphic function gh_ε on $\Omega := [-e^C \varepsilon^{-1}, e^C \varepsilon^{-1}] \times [0,1]$. We have

$$|g(z)h_\varepsilon(z)| \leqslant 1$$

for $z \in \partial\Omega$ and, by the maximum principle, this estimate holds also for $z \in \Omega$. Since also $|g(z)h_\varepsilon(z)| \leqslant 1$ outside Ω, we obtain finally that

$$|g(z)h_\varepsilon(z)| \leqslant 1 \quad \forall z \in \mathbf{R} \times [0,1],\ \varepsilon > 0.$$

Fixing $z \in \mathbf{R} \times [0,1]$ and considering the limit $\varepsilon \to 0$, we conclude that $|g(z)| = e^{\gamma(s,t)} \leqslant 1$, and therefore $\gamma(s,t) \leqslant 0$. Repeating the same argument with $-\gamma$ instead of γ, we obtain $\gamma(s,t) \geqslant 0$, so $\gamma \equiv 0$.

Hence β does not depend on t and is harmonic, which implies

$$\beta(s,t) = as + c$$

with real constants a and c. This completes the proof. □

3. Asymptotic behaviour of nondegenerate finite energy half-planes.

3.1. Convergence at infinity. Let (M, λ) be a closed 3-dimensional contact manifold, let $\mathscr{L} \subset M$ be a Legendrian knot, and let $x : [0, T] \to M$ be a nondegenerate characteristic chord; that is,

$$\dot{x}(t) = X_\lambda(x(t)) \quad \text{for } 0 < t < T,$$

$$x(0),\ x(T) \in \mathscr{L}$$

and

$$\ker \lambda(x(T)) = T\varphi_T(x(0))T_{x(0)}\mathscr{L} \oplus T_{x(T)}\mathscr{L},$$

where φ denotes the flow of the Reeb vector field X_λ.

For the moment, we stick to the case where

$$x(0) \neq x(T).$$

If it happens that x is a closed orbit, then we will only get an immersion in Lemma 3.2 below. It is possible to avoid this problem if we pass to the universal cover of a tubular neighbourhood of x and carry out our constructions there. In this section, we prove the following theorem.

THEOREM 3.1. *Let $\tilde{u}$ be a nonconstant finite energy half-plane. If $R_k \to \infty$ is a sequence of positive numbers so that $u(R_k e^{\pi i(t/T)})$ converges to a nondegenerate characteristic chord x, then we have, in fact,*

$$\lim_{R\to\infty} u(\mathrm{R}e^{\pi i(t/T)}) = x(t),$$

with convergence in $C^\infty([0, T])$.

First we show that there is some kind of normal form near $x([0, T])$.

LEMMA 3.2. *There are open neighbourhoods $U \subset M$ of $x([0, T])$, $V \subset \mathbf{R}^3$ of $\{(0, 0)\} \times [0, T]$, and a diffeomorphism $\psi : U \xrightarrow{\sim} V$ so that*

- *$\psi(x(t)) = (0, 0, t)$ for all $t \in [0, T]$,*
- *$\psi^*(dz + x\,dy) = \lambda|_U$.*

Proof. Take $\delta > 0$ so that $x : [-\delta, T + \delta] \to M$ is still an embedding (recall that we assume $x(0) \neq x(T)$). We identify $T_{x(0)}M = \ker \lambda(x(0)) \oplus \mathbf{R} \cdot X_\lambda(x(0))$ with $\mathbf{R}^3 = \mathbf{R}^2 \oplus \mathbf{R}$. If $B' \subset \mathbf{R}^3$ and $W \subset M$ are suitable open neighbourhoods of zero and $x(0)$, respectively, then the exponential map

$$\exp : B' \to W$$

with respect to any Riemannian metric on M is a diffeomorphism with $\exp(0) =$

$x(0)$ and $T\exp(0) = \mathrm{Id}_{\mathbf{R}^3}$. Then define

$$\Phi : B \times [-\delta, T+\delta] \to M$$

$$(x_1, x_2; t) \mapsto \varphi_t(\exp(x_1, x_2; 0)),$$

where $B = (\mathbf{R}^2 \times \{0\}) \cap B'$ and φ_t is the flow of X_λ. The map Φ has the following properties:

- $\Phi(0,0,t) = \varphi_t(x(0)) = x(t)$, in particular, $\Phi|_{\{(0,0)\}\times[-\delta,T+\delta]}$ is an embedding;
- the derivative

$$T\Phi(x_1, x_2, t) : \mathbf{R}^2 \times \mathbf{R} \to T_{\Phi(x_1,x_2,t)}M$$

  is an isomorphism if B is chosen sufficiently small.

Then there is a neighbourhood U of $\{(0,0)\} \times [-\delta, T+\delta]$, so that $\Phi|_U$ is a diffeomorphism onto its image. Moreover, we have the following with $\lambda_1 := \Phi^*\lambda$:

$$\lambda_1(0,0,t)(\xi_1, \xi_2, \tau) = (\varphi_t^*\lambda)(x(0)) \cdot \big(\tau X_\lambda(x(0)) + (\xi_1, \xi_2, 0)\big)$$

$$= \tau;$$

that is, $\lambda_1(0,0,t) = dt = \lambda_0(0,0,t)$ with $\lambda_0(x_1,x_2,t) = x_1\,dx_2 + dt$. We have used the fact that the flow of the Reeb vector field φ_t preserves the contact form $\varphi_t^*\lambda = \lambda$. The Reeb vector field of the contact form λ_1 is given by

$$X_{\lambda_1}(x_1, x_2, t) \equiv \frac{\partial}{\partial t}.$$

If we write $\lambda_1 = a\,dx_1 + b\,dx_2 + c\,dt$, then $d\lambda_1 = (\partial_{x_1} b - \partial_{x_2} a)\,dx_1 \wedge dx_2 + (\partial_{x_1} c - \partial_t a)\,dx_1 \wedge dt + (\partial_{x_2} c - \partial_t b)\,dx_2 \wedge dt$. We obtain from $X_{\lambda_1} \equiv (0,0,1)$ that c is identically 1 and

$$\partial_{x_1} c - \partial_t a \equiv -\partial_t a \equiv 0,$$

$$\partial_{x_2} c - \partial_t b \equiv -\partial_t b \equiv 0.$$

Hence

$$\lambda_1(x_1, x_2, t) = a(x_1, x_2)\,dx_1 + b(x_1, x_2)\,dx_2 + dt$$

and

$$d\lambda_1(x_1, x_2, t) = (\partial_{x_1} b - \partial_{x_2} a)\,dx_1 \wedge dx_2.$$

$\lambda_1 \wedge d\lambda_1$ never vanishes, and hence $\partial_{x_1} b - \partial_{x_2} a$ is never zero. We may assume that always $\partial_{x_1} b - \partial_{x_2} a > 0$, since otherwise we could have composed ψ with the diffeomorphism $(x_1, x_2, t) \mapsto (x_2, x_1, t)$, which would have interchanged the roles of a and b in the formulas above. Defining for $s \in [0, 1]$

$$\mu_s := (1 - s)\lambda_1 + s\lambda_0,$$

we compute

$$\begin{aligned} \mu_s \wedge d\mu_s &= [(1 - s)(\partial_{x_1} b - \partial_{x_2} a) + s]\, dx_1 \wedge dx_2 \wedge dt \\ &\neq 0. \end{aligned}$$

Hence $(\mu_s)_{0 \leqslant s \leqslant 1}$ is a family of contact forms all having the same Reeb vector field $\partial/\partial t = (0, 0, 1)$. We can write

$$\mathbf{R}^3 = \mathbf{R} \cdot (0, 0, 1) \oplus \ker \mu_s(x_1, x_2, t).$$

Since μ_s is a contact form, $d\mu_s$ must be nondegenerate on $\ker \mu_s$. Choose now a time-dependent vector field Y_s with

$$\begin{aligned} i_{Y_s} \mu_s &\equiv 0, \\ i_{Y_s}\, d\mu_s &= -\frac{d}{ds}\mu_s = \lambda_1 - \lambda_0. \end{aligned}$$

Choosing the neighbourhood near $\{(0,0)\} \times [-\delta, T + \delta]$ sufficiently small, the flow φ_s of Y_s exists until time one because λ_1 and λ_0 coincide on $\{(0,0)\} \times [-\delta, T + \delta]$, which implies $Y_s(0, 0, t) \equiv 0$ for all s. We compute

$$\begin{aligned} \frac{d}{ds}\varphi_s^* \mu_s &= \varphi_s^* \left(\frac{d}{ds}\mu_s + L_{Y_s}\mu_s \right) \\ &= \varphi_s^*(\lambda_0 - \lambda_1 + d(i_{Y_s}\mu_s) + i_{Y_s}\, d\mu_s) \\ &\equiv 0. \end{aligned}$$

Hence $\lambda_1 = \mu_0 = \varphi_0^* \mu_0 = \varphi_1^* \mu_1 = \varphi_1^*(dt + x_1\, dx_2)$, and $(\varphi_1 \circ \psi^{-1} \circ \phi^{-1})$ is the required diffeomorphism. □

Proof of Theorem 3.1. Assuming by Whitney's embedding theorem that M is embedded into some $\mathbf{R}^N$, we can equip the set $C^\infty([0, T], M)$ with the usual

Fréchet metric. Then the set

$$X := \{y \in C^\infty([0,T], M) \mid y(0), y(T) \in \mathscr{L}\}$$

also becomes a metric space.

By Lemma 3.2 and the nondegeneracy assumption, the following is true: There is an open neighbourhood $U \subset X$ of x so that U does not contain any path $y \in X$ that satisfies $\dot{y} = X_\lambda(y)$.

Define the set

$$\mathscr{S}_T := \{y \in X \mid \dot{y} = X_\lambda(y)\}.$$

Take open neighbourhoods $V_1, V_2 \subset X$ of x and $\mathscr{S}_T \setminus \{x\}$, respectively, which have positive distance from each other. As in the proof of Theorem 2.1, we consider, instead of $\tilde{u}$, the finite energy strip

$$\tilde{v} = (b, v) = \tilde{u} \circ \phi : \mathbf{R} \times [0, T] \to \mathbf{R} \times M$$

with $\phi(s,t) = e^{(\pi/T)(s+it)}$. By assumption, there is a sequence $s_k \to +\infty$ so that

$$v(s_k, .) \to x$$

in $C^\infty([0,T], M)$.

Hence $v(s_k, .) \in V_1$ for all large k. We now proceed indirectly. Assuming that $v(s,t)$ does not converge to $x(t)$ in $C^\infty([0,T], M)$ as $s \to \infty$, we pick a sequence $\sigma_k \to \infty$ so that $v(\sigma_k, t)$ does not converge to $x(t)$. By Theorem 2.1, the sequence (σ_k) has a subsequence (σ'_k) so that $v(\sigma'_k, .)$ converges to some $\tilde{x} \in \mathscr{S}_T$ in $C^\infty([0,T], M)$. Since $\tilde{x} \neq x$, we have $v(\sigma'_k, .) \in V_2$ for all large k. Passing to suitable subsequences of (s_k) and (σ'_k), we may assume that

$$s_k < \sigma'_k \leqslant s_{k+1}$$

for all k. Moreover, $s_k \in V_1$ and $\sigma'_k \in V_2$ for k sufficiently large.

Because $s \mapsto v(s, .)$ is a continuous path in X, we can choose $s'_k \in (s_k, \sigma_k)$ so that

$$v(s'_k, .) \notin V_1 \cup V_2. \tag{5}$$

Using Theorem 2.1 again, we conclude that (s'_k) has a subsequence (s''_k) so that $v(s''_k, .)$ converges to some $y \in \mathscr{S}_T$, which is not possible in view of $y \in V_1 \cup V_2$ and (5). □

3.2. Exponential decay estimates. We have shown in the last section that under the assumption of nondegeneracy, the finite energy half-plane actually converges asymptotically with the characteristic chord. Studying the finite energy half-plane carefully near the chord, we are able to show that this convergence is of exponential nature. In view of the local coordinates that we established in Lemma 3.2 and the convergence result (Theorem 3.1), we are in the following situation:

$$M = (\mathbf{R}^3, dz + x\,dy) = (\mathbf{R}^3, \lambda_0),$$

$$x_0(t) = (0, 0, t); \quad 0 \leqslant t \leqslant T.$$

For $k = 1, 2$, we have Legendrian curves

$$\gamma_k : (-1, +1) \to \mathbf{R}^3; \ \hat{\mathscr{L}}_k := \gamma_k((-1, +1)),$$

which are embedded and satisfy

$$\gamma_1(0) = (0, 0, 0),$$

$$\gamma_2(0) = (0, 0, T).$$

Since we assume that we have a nondegenerate situation, we get

$$\mathrm{Span}_{\mathbf{R}}\{\hat{e}_1 := \dot{\gamma}_1(0); \hat{e}_2 := \dot{\gamma}_2(0)\} = \mathbf{R}^2 \times \{0\}.$$

We have a finite energy strip

$$\tilde{v} : [s_0, \infty) \times [0, T] \to \mathbf{R} \times \mathbf{R}^3$$

$$\tilde{v} = (b, v) = (b; x, y, z),$$

where we choose s_0 so that the finite energy strip has an image in the coordinate neighbourhood given by Lemma 3.2 whenever $s \geqslant s_0$. We summarize some properties of v:

(1) $\partial_s b - \lambda_0(v)\partial_t v = 0$;

(2) $\lambda_0(v)\partial_s v + \partial_t b = 0$;

(3) $\pi_{\lambda_0}\partial_s v + j(v)\pi_{\lambda_0}\partial_t v = 0$, where $j : \ker \lambda_0 \to \ker \lambda_0$ is a complex structure compatible with $d\lambda_0$; that is, $j^2 = -\mathrm{Id}$ and $d\lambda_0 \circ (\mathrm{Id} \times j)$ is a bundle metric on $\ker \lambda_0$;

(4) $v(s, \cdot\,) \to x_0$ in $C^\infty([0, T], \mathbf{R}^3)$ as $s \to \infty$, which implies

$$\partial_t b(s, t) \xrightarrow[s \to \infty]{} 0,$$

$$\partial_s b(s, t) \xrightarrow[s \to \infty]{} 1$$

in $C^\infty([0, T], \mathbf{R}^3)$;

(5) $v(s, 0) \in \hat{\mathscr{L}}_1, v(s, T) \in \hat{\mathscr{L}}_2$.

The aim of this section is the following theorem, which states that we have exponential convergence of $\tilde{v}(s,t)$ to $(s,t) \mapsto (s + \text{const}, 0, 0, t)$.

THEOREM 3.3. *Let* $\tilde{v}: [s_0, \infty) \times [0, T] \to \mathbf{R} \times \mathbf{R}^3$ *be a finite energy strip as explained above. Then there are constants* $b_0 \in \mathbf{R}$, $r > 0, s_0 \in \mathbf{R}$ *so that for each multi-index* $\alpha = (\alpha_1, \alpha_2) \in \mathbf{N}^2$, $0 < \rho < \min\{r/2, 1/T\}$, *and* $s \geqslant s_0$ *we have*

$$\sup_{0 \leqslant t \leqslant T} |\partial^\alpha x(s,t)| \leqslant c_\alpha^1 e^{-rs},$$

$$\sup_{0 \leqslant t \leqslant T} |\partial^\alpha y(s,t)| \leqslant c_\alpha^1 e^{-rs},$$

$$\sup_{0 \leqslant t \leqslant T} |\partial^\alpha (z(s,t) - t)| \leqslant c_\alpha^2 e^{-\rho s},$$

$$\sup_{0 \leqslant t \leqslant T} |\partial^\alpha (b(s,t) - b_0 - s)| \leqslant c_\alpha^3 e^{-\rho\sigma},$$

where $\partial^\alpha = (\partial/\partial s)^{\alpha_1} (\partial/\partial t)^{\alpha_2}$ *and* $c_\alpha^1, c_\alpha^2, c_\alpha^3 > 0$ *are suitable constants. (We adopt the convention that zero is contained in* $\mathbf{N}$.*)*

Define a linear map $A_0 \in GL_3(\mathbf{R})$ by $A_0 \dot{\gamma}_1(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, A_0 \dot{\gamma}_2(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. If $\dot{\gamma}_1(0) = \begin{pmatrix} a_{11} \\ a_{21} \\ 0 \end{pmatrix}, \dot{\gamma}_2(0) = \begin{pmatrix} a_{12} \\ a_{22} \\ 0 \end{pmatrix}$, then $A_0^{-1} = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Considering $(b, A_0 v)$ instead of $(b, v) = \tilde{v}$, we may still assume that properties (1)–(5) hold with $\dot{\gamma}_1(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\dot{\gamma}_2(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and j a complex structure on $\ker \lambda_1$, where λ_1 is given by

$$\lambda_1 = (a_{11} x + a_{12} y) \cdot (a_{21}\, dx + a_{22}\, dy) + dz,$$

which is compatible with $d\lambda_1 = \det A^{-1} \cdot dx \wedge dy$. The Reeb vector field $X_{\lambda_1}(x, y, z)$ still equals $(0, 0, 1)$.

For $\delta > 0$ small, we may write

$$U_1 \cap \hat{\mathscr{L}}_1 = \{(x, f_1(x), g_1(x)) \mid x \in (-\delta, \delta)\},$$

$$U_2 \cap \hat{\mathscr{L}}_2 = \{f_2(y), y, g_2(y) \mid y \in (-\delta, \delta)\},$$

where U_1, U_2 are some open neighbourhoods of $(0, 0, 1)$ and $(0, 0, T)$, respectively, and f_1, f_2, g_1, g_2 are real-valued functions defined on $(-\delta, \delta)$ with

$$0 = f_1(0) = f_2(0),$$

$$0 = g_1(0); T = g_2(0),$$

$$0 = \dot{f}_1(0) = \dot{f}_2(0),$$

$$0 = \dot{g}_1(0) = \dot{g}_2(0).$$

Now choose a smooth function $\beta: \mathbf{R} \to [0,1]$ for some $0 < \varepsilon < T/2$ with

$$\begin{aligned} \beta &\equiv 0 \quad \text{on } (-\infty, \varepsilon) \\ \beta &\equiv 1 \quad \text{on } (T-\varepsilon, \infty) \\ \beta'(z) &\geqslant 0 \quad \text{for all } z \in \mathbf{R}. \end{aligned}$$

Define

$$\Phi : (-\delta,\delta) \times (-\delta,\delta) \times \mathbf{R} \to \mathbf{R}^3$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x - f_2(y) \\ y - f_1(x) \\ z + h(x,y,z) \end{pmatrix},$$

where h is given by

$$h(x,y,z) := \beta(z)\big[T - g_2(y) + g_1(x)\big] - g_1(x)$$

and where $\delta > 0$ is chosen so small that $\det D\Phi(x,y,z) \neq 0$ whenever $|x|$ and $|y|$ are smaller than δ. In view of $\Phi(0,0,z) = (0,0,z)$, the map Φ is a diffeomorphism between certain open neighbourhoods of $x_0([0,T]) = \{(0,0)\} \times [0,T]$ satisfying $D\Phi(0,0,z) = \mathrm{Id}_{\mathbf{R}^3}$. For sufficiently small δ, we have $\Phi(x, f_1(x), g_1(x)) \in \mathbf{R} \cdot (1,0,0)$ and $\Phi(f_2(y), y, g_2(y)) \in \mathbf{R} \cdot (0,1,0) + (0,0,T)$. For s large enough, we consider now $(b, \Phi(v))$ instead of $\tilde{v}$. The properties (1)–(5) still hold with λ_0 replaced by $\lambda_2 = (\Phi^{-1})^*\lambda_1$, while we may replace $\hat{\mathscr{L}}_1$ and $\hat{\mathscr{L}}_2$ by $\mathbf{R} \cdot (1,0,0)$ and $\mathbf{R} \cdot (0,1,0) + (0,0,T)$, respectively. Moreover, j is now a compatible complex structure on $\ker \lambda_2$. By computing $D\Phi$ and its inverse, we see that $\lambda_2(0,0,z) = dz$, $d\lambda_2(0,0,z) = \det A^{-1} \cdot dx \wedge dy$, and

$$X_{\lambda_2}(x,y,z) = (1 + \partial_z h(\Phi^{-1}(x,y,z)))^{-1} \cdot (0,0,1) =: f(x,y,z) \cdot (0,0,1)$$

with $f(0,0,z) \equiv 1$. Moreover, we have

$$\lambda_2 = \frac{1}{f} dz + c_1\, dx + c_2\, dy$$

with $c_1 \equiv c_2 \equiv 0$ on $\{(0,0)\} \times \mathbf{R}$ and $d\lambda_2 = a \cdot dx \wedge dy$ with $a \equiv \det A^{-1}$ on $\{(0,0)\} \times \mathbf{R}$. We calculate

$$df(p) = -(1 + \partial_z h(\Phi^{-1}(p)))^{-2} D\partial_z h(\Phi^{-1}(p)) D\Phi^{-1}(p);$$

hence

$$df(0,0,z) = -(D\partial_z h)(0,0,z) \equiv 0.$$

The contact plane $\ker \lambda_2(x,y,z)$ is generated by the vectors

$$e_1 := \left. \begin{pmatrix} 1 \\ 0 \\ -\dfrac{c_1}{1 + \partial_z h(\Phi^{-1})} \end{pmatrix} \right|_{(x,y,z)} = \left. \begin{pmatrix} 1 \\ 0 \\ -c_1 f \end{pmatrix} \right|_{(x,y,z)}$$

and

$$e_2 := \left. \begin{pmatrix} 0 \\ 1 \\ -\dfrac{c_2}{1 + \partial_z h(\Phi^{-1})} \end{pmatrix} \right|_{(x,y,z)} = \left. \begin{pmatrix} 0 \\ 1 \\ -c_2 f \end{pmatrix} \right|_{(x,y,z)}$$

We compute

$$\begin{aligned} \pi_{\lambda_2}(v_1, v_2, v_3) &= (v_1, v_2, v_3) - \frac{\lambda_2(v_1, v_2, v_3)}{1 + \partial_z h(\Phi^{-1})} \cdot (0,0,1) \\ &= v_1 e_1 + v_2 e_2. \end{aligned}$$

Then the equation $\pi_{\lambda_2} \partial_s v + j(v) \pi_{\lambda_2} \partial_t v = 0$ is equivalent to

$$\partial_s x \cdot e_1 + \partial_s y \cdot e_2 + j(x,y,z)(\partial_t x \cdot e_1 + \partial_t y \cdot e_2) = 0.$$

With

$$\begin{aligned} j(x,y,z) e_1 &= m_{11}(x,y,z) e_1 + m_{12}(x,y,z) e_2, \\ j(x,y,z) e_2 &= m_{21}(x,y,z) e_1 + m_{22}(x,y,z) e_2 \end{aligned}$$

and defining $M(x,y,z) := \begin{pmatrix} m_{11} & m_{21} \\ m_{12} & m_{22} \end{pmatrix}(x,y,z)$, we obtain

$$\begin{pmatrix} \partial_s x \\ \partial_s y \end{pmatrix} + M(x,y,z)\begin{pmatrix} \partial_t x \\ \partial_t y \end{pmatrix} = 0.$$

Summarizing, $\tilde{v} = (b,x,y,z) = (b,v)\colon [s_0,\infty) \times [0,T] \to \mathbf{R} \times \mathbf{R}^3$ satisfies the following:

(1′) $\partial_s b - \lambda_2(v)\partial_t v = 0$,

(2′) $\partial_t b + \lambda_2(v)\partial_s v = 0$,

(3′) $\begin{pmatrix} \partial_s x \\ \partial_s y \end{pmatrix} + M(x,y,z)\begin{pmatrix} \partial_t x \\ \partial_t y \end{pmatrix} = 0$,

(4′) $v(s,.) \to x_0$ in $C^\infty([0,T],\mathbf{R}^3)$ as $s \to \infty$, which implies

$$\partial_t b(s,t) \xrightarrow{s\to\infty} 0$$

$$\partial_s b(s,t) \xrightarrow{s\to\infty} 1$$

in $C^\infty([0,T],\mathbf{R}^3)$,

(5′) $v(s,0) \in \mathbf{R}\cdot(1,0,0)$ and $v(s,T) \in \mathbf{R}\cdot(0,1,0) + (0,0,T)$.

Since j is compatible with $d\lambda_2$, M has to satisfy

(a) $M^T J_0 M = J_0$ with $J_0 = \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}$,

(b) $-J_0 M > 0$.

We derive first an exponential decay estimate for the components $\zeta = (x,y)$ of v transversal to the orbit x_0 and to their derivatives. We write

$$\begin{aligned} 0 &= \partial_s\zeta(s,t) + M(\zeta(s,t),z(s,t))\partial_t\zeta(s,t) \\ &= \partial_s\zeta(s,t) + M(v(s,t))\partial_t\zeta(s,t). \end{aligned} \tag{6}$$

We consider the following family of inner products on $L^2([0,T],\mathbf{R}^2)$:

$$(u_1,u_2)_s := \int_0^T \langle u_1(t), -J_0 M(v(s,t))u_2(t)\rangle\, dt. \tag{7}$$

We define

$$W^{1,2}_\Gamma([0,T],\mathbf{R}^2) := \left\{ u \in W^{1,2}([0,T],\mathbf{R}^2) \,\middle|\, \begin{matrix} u(0) \in \mathbf{R}\cdot(1,0) \\ u(T) \in \mathbf{R}\cdot(0,1) \end{matrix} \right\},$$

where we use the inner products (7) to define

$$(u,v)_{s,1,2} := (u,v)_s + (u',v')_s.$$

We know that the matrices $M(v(s,t))$ converge to $M(0,0,t)$ as $s \to \infty$ uniformly in t. In particular, there is a positive constant C so that

$$|(-M(v(s,t))J_0)^{1/2}|,\, |(-J_0M(v(s,t)))^{1/2}| \leqslant C$$

for all $(s,t) \in [s_0, \infty) \times [0,T]$. Therefore the L^2-norms induced by the inner products (7) are equivalent to the ordinary L^2-norm $\|\,.\,\|_{L^2}$, and we have an estimate

$$C\|u\|_{L^2} \geqslant (u,u)_s^{1/2} \geqslant \frac{1}{C}\|u\|_{L^2}$$

with some constant $C > 0$ not depending on s. We prefer to use (7) instead of the ordinary L^2-product since it is better adapted to our problem, as we will soon see.

Define the following family of unbounded linear operators

$$A(s) : L^2([0,T],\mathbf{R}^2) \supset W_\Gamma^{1,2}([0,T],\mathbf{R}^2) \to L^2([0,T],\mathbf{R}^2)$$

$$(A(s)\cdot u)(t) := -M(v(s,t))\dot{u}(t).$$

Then (6) can be written as ($\zeta(s) := \zeta(s,.)$)

$$\partial_s\zeta(s)(t) = (A(s)\cdot\zeta(s))(t). \tag{8}$$

We state some basic properties of $A(s)$.

PROPOSITION 3.4. (1) *$A(s)$ is a selfadjoint operator on $(L^2([0,T],\mathbf{R}^2),\, (\cdot,\cdot)_s)$.*
(2) $\operatorname{Ker} A(s) = \{0\}$.
(3) *There is a constant $\delta > 0$, not depending on s, so that for all $\gamma \in W_\Gamma^{1,2}([0,T],\mathbf{R}^2)$ and $s \in [s_0,\infty)$,*

$$\|A(s)\cdot\gamma\|_s \geqslant \delta\|\gamma\|_s.$$

(In the following, we write $\|\,.\,\|_s := (\,.\,,\,.\,)_s^{1/2}$.)

Proof. First we note that $A(s)$ is densely defined since $C_0^\infty([0,T],\mathbf{R}^2)$ is contained in $W_\Gamma^{1,2}([0,T],\mathbf{R}^2)$ and is dense in $L^2([0,T],\mathbf{R}^2)$. We compute for $u_1, u_2 \in W_\Gamma^{1,2}([0,T],\mathbf{R}^2)$:

$$(A(s)u_1,u_2)_s = \int_0^T \langle -M(v(s,t))\dot{u}_1(t), -J_0M(v(s,t))u_2(t)\rangle\, dt$$

$$= \int_0^T \langle \dot{u}_1(t), J_0u_2(t)\rangle\, dt$$

$$= \left[\langle u_1(t), J_0 u_2(t)\rangle\right]_{t=0}^{t=T} - \int_0^T \langle u_1(t), J_0 \dot{u}_2(t)\rangle\, dt$$

$$= \int_0^T \langle J_0 u_1(t), \dot{u}_2(t)\rangle\, dt$$

$$= \int_0^T \langle (A(s)\cdot u_2)(t), M^T(v(s,t)) J_0 u_1(t)\rangle\, dt$$

$$= (u_1, A(s)u_2)_s.$$

Hence we have shown that $A(s)$ is symmetric, and therefore the adjoint operator $A^*(s)$ is an extension of $A(s)$. We have to prove that its domain of definition $D(A^*(s))$ is actually contained in $W_\Gamma^{1,2}([0,T],\mathbf{R}^2)$. We have $y \in D(A^*(s))$ if and only if there is some $y^* \in L^2([0,T],\mathbf{R}^2)$ so that

$$(A(s)x, y)_s = (x, y^*)_s \quad \text{for all } x \in W_\Gamma^{1,2}([0,T],\mathbf{R}^2).$$

Now let $y \in D(A^*(s))$. If $x \in C_0^\infty([0,T],\mathbf{R}^2)$, then

$$\begin{aligned}(\dot{x}, J_0 y)_{L^2} &= (A(s)x, y)_s \\ &= (x, y^*)_s \\ &= (x, -J_0 M(v(s))y^*)_{L^2}.\end{aligned}$$

Hence y has weak derivative $M(v(s))y^* \in L^2([0,T],\mathbf{R}^2)$, that is, $y \in W^{1,2}([0,T],\mathbf{R}^2)$. By the Sobolev multiplication theorem, the function

$$\varphi(t) := \langle x(t), -J_0 y(t)\rangle$$

is also in $W^{1,2}([0,T],\mathbf{R}^2)$ for any $x \in W_\Gamma^{1,2}([0,T],\mathbf{R}^2)$; and because of the Sobolev embedding theorem, φ is even continuous. Therefore

$$\begin{aligned}&\langle x(T), -J_0 y(T)\rangle - \langle x(0), -J_0 y(0)\rangle \\ &\qquad = \int_0^T \dot{\varphi}(t)\, dt \\ &\qquad = \int_0^T \langle \dot{x}(t), -J_0 y(t)\rangle\, dt + \int_0^T \langle x(t), -J_0 \dot{y}(t)\rangle\, dt \\ &\qquad = 0.\end{aligned}$$

If we pick now some $x \in W_{\Gamma}^{1,2}([0,T],\mathbf{R}^2)$ that satisfies $x(0)=(0,0)$ and $x(T) \neq (0,0)$, we conclude that $y(T) \in \mathbf{R}\cdot(0,1)$. Similarly, we get $y(0) \in$ m$\mathbf{R}\cdot(1,0)$ and we have shown $y \in D(A(s))$, so that $A(s)$ is selfadjoint.

Property (2) of Proposition 3.4 holds because there are no constant nonzero paths in $W_{\Gamma}^{1,2}([0,T],\mathbf{R}^2)$. Assume now that (3) is false. Then for each sequence $\delta_k \searrow 0$, there are sequences $(\gamma_k) \subseteq W_{\Gamma}^{1,2}([0,T],\mathbf{R}^2)$ and $(s_k) \subseteq [s_0,\infty)$ so that

$$\|A(s_k)\cdot\gamma_k\|_{s_k} < \delta_k\|\gamma_k\|_{s_k}.$$

Defining $\alpha_k := \gamma_k/\|\gamma_k\|_{s_k} \in W_{\Gamma}^{1,2}([0,T],\mathbf{R}^2)$, we obtain

$$\|\dot{\alpha}_k\|_{L^2} \leqslant C\|\dot{\alpha}_k\|_{s_k} = C\|A(s_k)\alpha_k\|_{s_k} < C\delta_k \searrow 0$$

$$\|\alpha_k\|_{L^2} \leqslant C\|\alpha_k\|_{s_k} = C$$

$$\|\alpha_k\|_{L^2} \geqslant \frac{1}{C}\|\alpha_k\|_{s_k} = \frac{1}{C} > 0.$$

Since the inclusion $W^{1,2}([0,T],\mathbf{R}^2) \hookrightarrow L^2([0,T],\mathbf{R}^2)$ is compact, we may pass to some subsequence (which we still denote by (α_k)) that converges in L^2 to some $\alpha \in L^2([0,T],\mathbf{R}^2)$. Then (α_k) converges in $W^{1,2}$ to α and $\dot{\alpha}=0$, so α must be constant. Recalling that $\alpha_k \in W_{\Gamma}^{1,2}([0,T],\mathbf{R}^2)$ and that $W_{\Gamma}^{1,2}$ is a closed subspace of $W^{1,2}([0,T],\mathbf{R}^2)$, we obtain $\alpha \in W_{\Gamma}^{1,2}([0,T],\mathbf{R}^2)$ as well; therefore, $\alpha = 0$ which contradicts

$$\|\alpha_k\|_{L^2} \geqslant \frac{1}{C} > 0,$$

and we are done. □

For fixed $l' \in \mathbf{N}$ and $\zeta(s) \in W_{\Gamma}^{l'+2,2}([0,T],\mathbf{R}^2)$, we introduce the column vector

$$W(s) := \left(\zeta(s), \frac{\partial}{\partial s}\zeta(s), \dots, \frac{\partial^{l'}}{\partial s^{l'}}\zeta(s)\right).$$

We note that each component of $W(s)$ satisfies the boundary conditions

$$\frac{\partial^k}{\partial s^k}\zeta(s,0) \in \mathbf{R}\cdot(1,0) \quad \text{and} \quad \frac{\partial^k}{\partial s^k}\zeta(s,T) \in \mathbf{R}\cdot(0,1).$$

Hence we can view $W(s)$ as an element in $W_{\Gamma}^{2,2}([0,T],\mathbf{R}^{2(l'+1)})$.

Applying $\partial_s^k := \partial^k/\partial s^k$ to our differential equation (8) (with $k \geqslant 1$), we get

$$\begin{aligned}\partial_s(\partial_s^k\zeta(s))(t) &= (A(s)\cdot\partial_s^k\zeta(s))(t) + \sum_{l=1}^{k}\binom{k}{l}\partial_s^l(-M(v(s,t)))\partial_t(\partial_s^{k-l}\zeta(s,t))\\ &=: (A(s)\cdot\partial_s^k\zeta(s))(t) + \sum_{l=1}^{k}\Delta_{lk}(s,t)\cdot\partial_t(\partial_s^{k-l}\zeta(s,t)).\end{aligned}$$

If we put

$$\begin{aligned}\tilde{M}(v(s,t)) &:= \begin{pmatrix} M(v(s,t)) & 0 & \dots & 0\\ 0 & & & \vdots\\ \vdots & & \ddots & \vdots\\ \vdots & & & 0\\ 0 & \dots & 0 & M(v(s,t))\end{pmatrix}\\ &= \operatorname{diag}(M(v(s,t)),\dots,M(v(s,t))),\\ \tilde{A}(s) &:= -\tilde{M}(v(s,\,.\,))\cdot\frac{\partial}{\partial t},\\ \hat{\Delta}(s,t) &:= \begin{pmatrix} 0 & 0 & 0 & \dots & \dots & 0\\ \Delta_{11}(s,t) & 0 & 0 & \dots & \dots & 0\\ \Delta_{22}(s,t) & \Delta_{12}(s,t) & 0 & \dots & \dots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ \Delta_{l'l'}(s,t) & \Delta_{l'-1,l'}(s,t) & \Delta_{l'-2,l'}(s,t) & \cdots & \Delta_{1,l'}(s,t) & 0\end{pmatrix},\end{aligned}$$

then we obtain an equation for $W(s)$:

$$\partial_s W(s) = \tilde{A}(s)\cdot W(s) + \hat{\Delta}(s,\,.\,)\partial_t W(s). \tag{9}$$

Remark. Let us emphasize a tiny detail that becomes extremely important later: Although $W(s)$ contains derivatives $\partial_s^k\zeta$ up to order $k = l'$, the expression $\hat{\Delta}(s,\,.\,)\partial_t W(s)$ only contains s-derivatives up to order $l'-1$ because $\partial_s^{l'}\zeta$ appears already in $\tilde{A}(s)\cdot W(s)$.

We define the nonnegative function

$$g(s) := \frac{1}{2}\|W(s)\|_s^2 = \frac{1}{2}\int_0^T \langle W(s,t), -\tilde{J}_0\tilde{M}(v(s,t))W(s,t)\rangle\, dt,$$

where $\tilde{J}_0 = \mathrm{diag}(J_0, \dots, J_0)$. Then

$$g'(s) = (\partial_s W(s), W(s))_s + \frac{1}{2}\int_0^T \langle W(s,t), -\tilde{J}_0\partial_s\tilde{M}(v(s,t))W(s,t)\rangle\, dt$$

and

$$\begin{aligned} g''(s) &= (\partial_{ss}W(s), W(s))_s + \|\partial_s W(s)\|_2^2 \\ &\quad + 2\int_0^T \langle \partial_s W(s,t), -\tilde{J}_0\partial_s[\tilde{M}(v(s,t))]W(s,t)\rangle\, dt \\ &\quad + \frac{1}{2}\int_0^T \langle W(s,t), -\tilde{J}_0\partial_{ss}[\tilde{M}(v(s,t))]W(s,t)\rangle\, dt \\ &=: T_1 + T_2 + T_3 + T_4 \\ &\geqslant T_1 + T_3 + T_4. \end{aligned}$$

Note. In the following, we write $\varepsilon(s)$ for a positive function that converges to zero as $s \to \infty$, if it does not matter what $\varepsilon(s)$ actually is. We also denote positive constants by c if the size of the constant is not important.

We estimate T_3 and T_4 as follows, with $\| \,.\, \|_{L^2}$ being the ordinary L^2-norm:

$$\begin{aligned} |T_3| &\leqslant \varepsilon(s)\|\partial_s W(s)\|_{L^2}\|W(s)\|_{L^2} \\ &\leqslant \varepsilon(s)(1 + |\hat{\Delta}(s)\tilde{M}(v(s))|_{C^0([0,T])})\|\tilde{A}(s)\cdot W(s)\|_{L^2}\|W(s)\|_{L^2} \\ &\leqslant \varepsilon(s)\|\tilde{A}(s)\cdot W(s)\|_s\|W(s)\|_s, \\ |T_4| &\leqslant \varepsilon(s)\|W(s)\|_{L^2}^2 \\ &\leqslant \varepsilon(s)\|W(s)\|_s^2. \end{aligned}$$

We obtain this from the fact that $v(s,t) \to (0,0,t)$. This implies also

$$|\hat{\Delta}(s,t)|, |\partial_s\hat{\Delta}(s,t)| \to 0 \tag{10}$$

as $s \to \infty$ uniformly in t.

Now we estimate T_1. We calculate

$$\partial_{ss}W(s) = -\partial_s[\tilde{M}(v(s))]\partial_t W(s) + \tilde{A}(s)\cdot\partial_s W(s) + \partial_s\hat{\Delta}(s)\partial_t W(s) + \hat{\Delta}(s)\partial_{st}W(s)$$

and

$$\begin{aligned}(\partial_{ss}W(s), W(s))_s &= (-\partial_s[\tilde{M}(v(s))]\tilde{M}(v(s))(\tilde{A}(s)\cdot W(s)), W(s))_s \\ &\quad + (\tilde{A}(s)\cdot\partial_s W(s), W(s))_s \\ &\quad + (\partial_s\hat{\Delta}(s)\partial_t W(s), W(s))_s + (\hat{\Delta}(s)\partial_{st}W(s), W(s))_s \\ &=: T_5 + T_6 + T_7 + T_8.\end{aligned}$$

We begin with T_5:

$$|T_5| \leqslant \varepsilon(s)\|\tilde{A}(s)\cdot W(s)\|_s\|W(s)\|_s.$$

Then

$$\begin{aligned}T_6 &= (\tilde{A}(s)\cdot\partial_s W(s), W(s))_s \\ &= (\partial_s W(s), \tilde{A}(s)\cdot W(s))_s \quad \text{(by Proposition 3.4)} \\ &= \|\tilde{A}(s)\cdot W(s)\|_s^2 + (\hat{\Delta}(s)\partial_t W(s), \tilde{A}(s)\cdot W(s))_s \\ &= \|\tilde{A}(s)\cdot W(s)\|_s^2 + (\hat{\Delta}(s)\tilde{M}(v(s))(\tilde{A}(s)\cdot W(s)), \tilde{A}(s)\cdot W(s))_s \\ &\geqslant \|\tilde{A}(s)\cdot W(s)\|_s^2 - \varepsilon(s)\|\tilde{A}(s)\cdot W(s)\|_s^2 \\ &\geqslant \frac{1}{2}\|\tilde{A}(s)\cdot W(s)\|_s^2,\end{aligned}$$

if s is chosen large enough. Next

$$\begin{aligned}|T_7| &\leqslant |(\partial_s\hat{\Delta}(s)\tilde{M}(v(s))(\tilde{A}(s)\cdot W(s)), W(s))_s| \\ &\leqslant \varepsilon(s)\|\tilde{A}(s)\cdot W(s)\|_s\|W(s)\|_s.\end{aligned}$$

We are left with T_8. Putting

$$\tilde{\Delta}(s,t) := \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & \Delta_{11}(s,t) & 0 & \dots & 0 \\ 0 & \Delta_{22}(s,t) & \Delta_{12}(s,t) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \Delta_{l',l'}(s,t) & \Delta_{l'-1,l'}(s,t) & \dots & \Delta_{1,l'}(s,t) \end{pmatrix},$$

we see that

$$\hat{\Delta}(s)\partial_{st}W(s) = \tilde{\Delta}(s)\partial_t W(s) = \tilde{\Delta}(s)\tilde{M}(v(s))(\tilde{A}(s)\cdot W(s)). \tag{11}$$

Remark. It is not a mistake that the derivative $\partial_s^{l'+1}\zeta$ seems to disappear in the above equation. The reason for this "miracle" is simply that $\partial_s^{l'+1}\zeta$ is not even contained in the left-hand side of (11); see also the remark after equation (9).

Therefore

$$\begin{aligned} |T_8| &\leqslant |(\hat{\Delta}(s,\,.\,)\partial_{st}W(s), W(s))_s| \\ &\leqslant \varepsilon(s)\|\tilde{A}(s)\cdot W(s)\|_s\|W(s)\|_s. \end{aligned}$$

Summarizing, we finally get

$$\begin{aligned} g''(s) &\geqslant \frac{1}{2}\|\tilde{A}(s)\cdot W(s)\|_s^2 - \varepsilon(s)\|\tilde{A}(s)\cdot W(s)\|_s\|W(s)\|_s \\ &= \|\tilde{A}(s)\cdot W(s)\|_s\left(\frac{1}{2}\|\tilde{A}(s)\cdot W(s)\|_s - \varepsilon(s)\|W(s)\|_s\right) \\ &\geqslant \left(\frac{\delta}{2} - \varepsilon(s)\right)\|\tilde{A}(s)\cdot W(s)\|_s\|W(s)\|_s, \end{aligned}$$

by Proposition 3.4. The term in the bracket is always larger than $\delta/4$ if s is sufficiently large; hence

$$g''(s) \geqslant \frac{\delta^2}{4}\|W(s)\|_s^2 = \frac{\delta^2}{2}g(s) \tag{12}$$

using Proposition 3.4 again. Because we can estimate $\|\,.\,\|_s$ from above by the ordinary L^2-norm and $|\partial_s^k\zeta(s,\,.\,)| \to 0$ in $C^0([0,T],\mathbf{R}^2)$ as $s\to\infty$, we conclude that

$$0 \leqslant g(s) \to 0 \quad \text{as } s \to 0.$$

Then (12) implies

$$g(s) \leqslant g(s_0)e^{-(\delta/\sqrt{2})(s-s_0)} \quad \text{for all } s \geqslant s_0 \tag{13}$$

if s_0 is chosen large enough. Inequality (13) implies that for some suitable s_0 and each $k \in \mathbf{N}$, there is a constant $c_k > 0$ so that

$$\|\partial_s^k \zeta(s)\|_{L^2} \leqslant c_k e^{-rs} \quad \forall s \geqslant s_0, \tag{14}$$

where $r = \delta/\sqrt{2}$.

We estimate with suitable $c > 0$, $0 < \varepsilon(s) \to 0$ using (10):

$$\begin{aligned} \|\partial_t W(s)\|_s &= \|\tilde{A}(s) \cdot W(s)\|_s \\ &\leqslant \|\partial_s W(s)\|_s + \|\hat{\Delta}(s)\partial_t W(s)\|_s \\ &\leqslant ce^{-rs} + \varepsilon(s)\|\partial_t W(s)\|_s. \end{aligned}$$

Hence

$$\|\partial_t \partial_s^k \zeta(s)\|_{L^2} \leqslant c_k' e^{-rs} \quad \text{for } s \geqslant s_0$$

with some constants c_k' and $k \geqslant 0$. Using $\partial_t \zeta(s,t) = M(v(s,t))\partial_s \zeta(s,t)$, we compute for $k, m \geqslant 1$,

$$\partial_t^{k+1}\partial_s^m \zeta(s,t) = \sum_{n=0}^{k}\sum_{l=0}^{m} \binom{k}{n}\binom{m}{l} \partial_t^{k-n}\partial_s^{m-l} M(v(s,t)) \cdot \partial_t^n \partial_s^{l+1} \zeta(s,t).$$

Inductively, we get estimates

$$\|\partial^\alpha \zeta(s)\|_{L^2} \leqslant c_\alpha e^{-rs} \quad \text{for } s \geqslant s_0,$$

where $\alpha = (\alpha_1, \alpha_2) \in \mathbf{N}^2$ and $\partial^\alpha = (\partial/\partial s)^{\alpha_1}(\partial/\partial t)^{\alpha_2}$. Applying the Sobolev embedding theorem, we obtain, finally,

$$|\partial^\alpha \zeta(s)|_{C^0([0,T],\mathbf{R}^2)} \leqslant c_\alpha e^{-rs} \quad \forall s \geqslant s_0.$$

We use this estimate now to derive also exponential decay estimates for $\partial^\alpha(z(s,t) - t)$ and $\partial^\alpha(b(s,t) - b_0 - s)$, which completes the proof of our theorem.

Let us study equations (1) and (2) now:

$$0 = \partial_s b - \frac{1}{f(v)}\partial_t z - c_1(v)\partial_t x - c_2(v)\partial_t y,$$

$$0 = f(v)\partial_t b + \partial_s z + f(v)c_1(v)\partial_s x + f(v)c_2(v)\partial_s y.$$

Recalling that $f(0,0,z) \equiv 1$, we may write

$$f(v(s,t)) = 1 + \int_0^1 df(\sigma\zeta(s,t), z(s,t)) \cdot \zeta(s,t)\, d\sigma =: 1 + F_1(s,t) \cdot \zeta(s,t)$$

and

$$\frac{1}{f(v(s,t))} = 1 + \int_0^1 -\left(f(\sigma\zeta(s,t), z(s,t))\right)^{-2} df(\sigma\zeta(s,t), z(s,t)) \cdot \zeta(s,t)\, d\sigma$$

$$=: 1 + F_2(s,t) \cdot \zeta(s,t).$$

We obtain the following equation:

$$\partial_s w + J_0 \partial_t w = h,$$

where

$$w(s,t) := \begin{pmatrix} w_1(s,t) \\ w_2(s,t) \end{pmatrix} =: \begin{pmatrix} b(s,t) - s \\ z(s,t) - t \end{pmatrix}.$$

Here $J_0 = \left(\begin{smallmatrix} 0 & -1 \\ +1 & 0 \end{smallmatrix}\right)$ and

$$h(s,t) := \begin{pmatrix} \partial_t z(F_2 \cdot \zeta) + c_1(v)\partial_t x + c_2(v)\partial_t y \\ -\partial_t b(F_1 \cdot \zeta) - (1 + F_1 \cdot \zeta)\left[c_1(v)f(v)\partial_s x + c_2(v)f(v)\partial_s y\right] \end{pmatrix}(s,t).$$

We know already that there are constants $s_0 \in \mathbf{R}$; $c', r > 0$ so that

$$|\zeta(s,\,.\,)|_{C^0([0,T])}, |\partial_s\zeta(s,\,.\,)|_{C^0([0,T])}, |\partial_t\zeta(s,\,.\,)|_{C^0([0,T])} \leqslant c' e^{-rs}$$

if $s \geqslant s_0$, while the other expressions involved in the definition of h remain bounded if $s \to \infty$. Hence

$$|h(s,\,.\,)|_{C^0([0,T],\mathbf{R}^2)} \leqslant c e^{-rs}$$

if $s \geqslant s_0$, where $c > 0$ is a suitable constant. By the same reasoning, all derivatives $\partial^\alpha h$ of h also satisfy an estimate as above with some constant $c_\alpha > 0$ depending on α.

We fix some number $l' \in \mathbf{N}$ and introduce the column vectors

$$V := B + iZ := (w_1 + iw_2, \partial_s w_1 + i\partial_s w_2, \dots, \partial_s^{l'} w_1 + i\partial_s^{l'} w_2),$$

$$H := (h, \partial_s h, \dots, \partial_s^{l'} h).$$

Then

$$\partial_s V + i\,\partial_t V = H,$$

where $i = \operatorname{diag}(J_0, \dots, J_0)$, and H satisfies

$$|\partial^\alpha H(s)|_{C^0([0,T])} \leqslant c_{\alpha,l'} e^{-rs}$$

for $s \geqslant s_0$, where $\alpha \in \mathbf{N}^2$ is some multi-index and $c_{\alpha,l'} > 0$ is a constant.

Moreover, we have the following boundary condition:

$$Z(s, 0) \equiv Z(s, T) \equiv 0.$$

Writing $H = H_1 + iH_2$, we obtain

$$\partial_s B - \partial_t Z = H_1$$

and

$$\int_0^T \partial_s B(s,t)\,dt = \int_0^T H_1(s,t)\,dt,$$

hence

$$\left|\int_0^T \partial_s B(s,t)\,dt\right| \leqslant ce^{-rs}.$$

We estimate

$$\begin{aligned}\left|\int_0^T B(s_2,t)\,dt - \int_0^T B(s_1,t)\,dt\right| &= \left|\int_{s_1}^{s_2}\int_0^T \partial_s B(\sigma,t)\,dt\,d\sigma\right| \\ &\leqslant \int_{s_1}^{s_2}\left|\int_0^T \partial_s B(\sigma,t)\,dt\right| d\sigma \qquad (15) \\ &\leqslant \frac{c}{r}(e^{-rs_1} - e^{-rs_2}).\end{aligned}$$

Hence $\int_0^T B(s,t)\,dt \to B_0$ for a suitable $B_0 \in \mathbf{R}^{l'+1}$ as $s \to \infty$. Considering $B(s,t) - (B_0/T)$ instead of $B(s,t)$, we may assume that the mean value $\int_0^T B(s,t)\,dt$ of B converges to zero for $s \to \infty$.

Taking the limit $s_2 \to \infty$ in (15), we get

$$\left|\int_0^T B(s,t)\,dt\right| \leqslant \frac{c}{r} e^{-rs}.$$

We define now

$$\bar{V}(s,t) := V(s,t) - \frac{1}{T}\int_0^T B(s,t)\,dt$$

$$= B(s,t) - \frac{1}{T}\int_0^T B(s,t)\,dt + iZ(s,t).$$

If we succeed in getting exponential decay estimates for $\bar{V}$, then we also have them for V since the mean value of the real part B decays already exponentially, as we saw above.

$\bar{V}$ satisfies the differential equation

$$\partial_s \bar{V} + i\,\partial_t \bar{V} = H_1 - \frac{1}{T}\int_0^T H_1\,dt + iH_2 =: \bar{H}_1 + i\bar{H}_2 =: \bar{H},$$

where $\bar{H}$ still decays like e^{-rs}. The real part $\bar{B}$ of $\bar{V}$ has mean value zero with respect to t, and the imaginary part vanishes on $[s_0, \infty) \times \{0, T\}$.

We begin with the following lemma.

LEMMA 3.5. *Assume* $v\colon [s_0,\infty) \times [0,T] \to \mathbf{C}^N \approx \mathbf{R}^{2N}$ *is bounded and solves*

(a) $\partial_s v + i\,\partial_t v = H$

with

(b) $|H(s)|_{C^0([0,T])} \leqslant ce^{-rs}$ $(r, c > 0)$

and

(c) $\partial_t v(s,.) \to 0$ *in* $C^0([0,T])$ *as* $s \to \infty$.

If $v = v_1 + i\,v_2$, *we further assume*

(d) $\int_0^T v_1(s,t)\,dt = 0$,

(e) $v_2(s,0) \equiv v_2(s,T) \equiv 0$.

Then

$$\int_{s_0}^{\infty} e^{2\rho s}\|v(s)\|_{L^2}^2\,ds \leqslant \int_{s_0}^{\infty} e^{2\rho s}\|\partial_t v(s)\|_{L^2}^2\,ds < \infty \tag{16}$$

for all $0 \leqslant \rho < \min\{r/2, 1/T\}$.

Proof. We have the following simple pointwise identities for a function $f : \mathbf{C} \to \mathbf{C}$ (identifying $\mathbf{C}$ with $\mathbf{R}^2$ via $z = s + it \mapsto (s,t)$):

(1) $d/ds\langle f, i\partial_t f\rangle - d/dt\langle f, i\partial_s f\rangle = 2\langle \partial_s f, i\partial_t f\rangle$,

(2) $|\partial_s f + i\partial_t f|^2 = |\partial_s f|^2 + |\partial_t f|^2 + 2\langle \partial_s f, i\partial_t f\rangle$.

First, we show that $s \mapsto \|v(s)\|_{L^2([0,T])}$ is in $L^2([s_0,\infty))$. We take

$$v_1(s,t) - v_1(s,t') = \int_{t'}^{t} \partial_t v_1(s,\tau)\, d\tau$$

and integrate with respect to t':

$$v_1(s,t) = \frac{1}{T}\int_0^T \int_{t'}^{t} \partial_t v_1(s,\tau)\, d\tau\, dt'.$$

This implies

$$\begin{aligned} |v_1(s,t)| &\leqslant \frac{1}{T}\int_0^T \int_0^T |\partial_t v_1(s,\tau)|\, d\tau\, dt' \\ &= \int_0^T |\partial_t v_1(s,t)|\, d\tau \\ &\leqslant T^{1/2}\|\partial_t v_1(s)\|_{L^2}. \end{aligned}$$

On the other hand,

$$v_2(s,t) = \int_0^t \partial_t v_2(s,\tau)\, d\tau,$$

hence

$$|v_2(s,t)| \leqslant \int_0^T |\partial_t v_2(s,t)|\, dt \leqslant T^{1/2}\|\partial_t v_2(s)\|_{L^2},$$

and therefore

$$\|v(s)\|_{L^2}^2 = \|v_1(s)\|_{L^2}^2 + \|v_2(s)\|_{L^2}^2 \leqslant T^2\|\partial_t v(s)\|_{L^2}^2. \tag{17}$$

We estimate

$$\begin{aligned}\int_{s_0}^{s} \|v(s')\|_{L^2}^2 &\leqslant T^2 \int_{s_0}^{s} \|\partial_t v(s')\|_{L^2}^2 \, ds' \\ &\leqslant T^2 \int_{s_0}^{s} (\|\partial_t v(s')\|_{L^2}^2 + \|\partial_s v(s')\|_{L^2}^2) \, ds' \\ &= T^2 \int_{s_0}^{s} \|H(s')\|_{L^2}^2 \, ds' \\ &\quad - 2T^2 \int_{s_0}^{s} (\partial_s v(s'), i\partial_t v(s'))_{L^2} \, ds' \\ &= T^2 \int_{s_0}^{s} \|H(s')\|_{L^2}^2 \, ds' \\ &\quad - T^2 \int_{s_0}^{s} \int_0^T \frac{d}{ds'} \langle v(s',t), i\partial_t v(s',t)\rangle \, dt \, ds' \\ &\quad + T^2 \int_{s_0}^{s} \int_0^T \frac{d}{dt} \langle v(s',t), i\partial_s v(s',t)\rangle \, dt \, ds' \\ &= T^2 \int_{s_0}^{s} \|H(s')\|_{L^2}^2 \, ds' + T^2 (v(s_0), i\partial_t v(s_0))_{L^2} \\ &\quad - T^2 (v(s), i\partial_t v(s))_{L^2}.\end{aligned}$$

We have used

$$\int_0^T \frac{d}{dt} \langle v(s',t), i\partial_s v(s',t)\rangle \, dt = \langle v(s',T), i\partial_s v(s',T)\rangle - \langle v(s',0), i\partial_s v(s',0)\rangle.$$

The right-hand side is equal to zero because we know that v, $\partial_s v|_{[s_0,\infty)\times\{0,T\}} \subseteq \mathbf{R}^N \subset \mathbf{C}^N$ in view of the boundary condition. Finally,

$$|(v(s), i\partial_t v(s))_{L^2}| \leqslant \text{const} \cdot \|\partial_t v(s)\|_{L^2} \longrightarrow 0$$

as $s \longrightarrow \infty$, so we get in the limit $s \longrightarrow \infty$:

$$\int_{s_0}^{\infty} \|v(s)\|_{L^2}^2 \, ds \leqslant T^2 \int_{s_0}^{\infty} \|H(s)\|_{L^2}^2 \, ds + T^2 (v(s_0), i\partial_t v(s_0))_{L^2} < \infty.$$

For $n \geqslant s_0$, we pick functions $\gamma_n : [0, \infty) \to [0, \infty)$ with

$$\begin{aligned} \gamma_n(s) &= s && \text{for } 0 \leqslant s \leqslant n, \\ \gamma_n(s) &\equiv \text{const} && \text{for } s \geqslant n+1, \\ 0 \leqslant \gamma_n'(s) &\leqslant 1 && \text{for all } 0 \leqslant s < \infty \text{ as } n \to \infty, \\ \gamma_n(s) &\nearrow s. \end{aligned}$$

Define

$$v_n(s,t) := e^{\rho\gamma_n(s)} v(s,t).$$

Then also $\|v_n(s)\| \in L^2([s_0, \infty))$, $\partial_t v_n(s,t) \to 0$ as $s \to \infty$ uniformly in t, the real part of v_n has mean value zero, and the imaginary part vanishes on $[s_0, \infty) \times \{0, T\}$.

We compute

$$\begin{aligned} \partial_s v_n + i\partial_t v_n &= \rho\gamma_n' e^{\rho\gamma_n} v + e^{\rho\gamma_n}\partial_s v + ie^{\rho\gamma_n}\partial_t v \\ &= \rho\gamma_n' v_n + e^{\rho\gamma_n} H. \end{aligned}$$

As before, we can estimate

$$\begin{aligned} \int_{s_0}^{\infty} \|v_n(s)\|_{L^2}^2\, ds &\leqslant T^2 \int_{s_0}^{\infty} \|\partial_t v_n(s)\|_{L^2}^2\, ds \\ &\leqslant T^2 \int_{s_0}^{\infty} \|\rho\gamma_n'(s) v_n(s) + e^{\rho\gamma_n(s)} H(s)\|_{L^2}^2\, ds \\ &\quad + T^2 (v_n(s_0), i\partial_t v_n(s_0))_{L^2} \\ &= T^2 \int_{s_0}^{\infty} \rho^2 (\gamma_n'(s))^2 \|v_n(s)\|_{L^2}^2\, ds \\ &\quad + T^2 \int_{s_0}^{\infty} e^{2\rho\gamma_n(s)} \|H(s)\|_{L^2}^2\, ds \\ &\quad + T^2 \int_{s_0}^{\infty} \rho\gamma_n'(s) e^{2\rho\gamma_n(s)} (v(s), H(s))_{L^2}\, ds \\ &\quad + T^2 e^{2\rho\gamma_n(s_0)} (v(s_0), i\partial_t v(s_0))_{L^2}. \end{aligned}$$

Recalling that $\|H(s)\|_{L^2}^2 \leqslant c^2 T e^{-2rs}$, we obtain

$$\int_{s_0}^{\infty} \|v_n(s)\|_{L^2}^2 \, ds \leqslant T^2 \int_{s_0}^{\infty} \|\partial_t v_n(s)\|_{L^2}^2 \, ds$$

$$\leqslant T^2 \rho^2 \int_{s_0}^{\infty} \|v_n(s)\|_{L^2}^2 \, ds + c^2 T^3 \int_{s_0}^{\infty} e^{2(\rho - r)s} \, ds$$

$$+ \tilde{c} T^{5/2} \rho \int_{s_0}^{\infty} e^{(2\rho - r)s} \, ds + T^2 e^{2\rho s_0} (v(s_0), i\partial_t v(s_0))_{L^2},$$

and therefore

$$\int_{s_0}^{\infty} \|v_n(s)\|_{L^2}^2 \, ds \leqslant T^2 \int_{s_0}^{\infty} \|\partial_t v_n(s)\|_{L^2}^2 \, ds$$

$$\leqslant T^2 \rho^2 \int_{s_0}^{\infty} \|v_n(s)\|_{L^2}^2 \, ds + M,$$

with some number $M > 0$ not depending on n. Since $\rho < T^{-1}$, this implies

$$\int_{s_0}^{\infty} \|v_n(s)\|_{L^2}^2 \, ds \leqslant \frac{M}{1 - T^2 \rho^2} < \infty$$

and

$$\int_{s_0}^{\infty} \|v_n(s)\|_{L^2}^2 \, ds \leqslant T^2 \int_{s_0}^{\infty} \|\partial_t v_n(s)\|_{L^2}^2 \, ds$$

$$\leqslant M \frac{1 + T^2 \rho^2}{1 - T^2 \rho^2}$$

$$< \infty.$$

In the limit as $n \to \infty$, we finally get

$$\int_{s_0}^{\infty} e^{2\rho s} \|v(s)\|_{L^2}^2 \, ds \leqslant T^2 \int_{s_0}^{\infty} e^{2\rho s} \|\partial_t v(s)\|_{L^2}^2 \, ds < \infty,$$

which completes the proof of Lemma 3.5. □

Applying this to $\bar{V}$, we find

$$\int_{s_0}^{\infty} e^{2\rho s} \|\bar{V}(s)\|_{L^2}^2 \, ds \leqslant T^2 \int_{s_0}^{\infty} e^{2\rho s} \|\partial_t \bar{V}(s)\|_{L^2}^2 \, ds < \infty$$

whenever $0 \leqslant \rho < \min\{T^{-1}, r/2\}$. Now we want to show that for each $\delta > 0$, there is a sequence $s_k \to \infty$ with $s_{k+1} - s_k \leqslant \delta$ so that $e^{\rho s_k}\bar{V}(s_k, t)$ converges to zero as $k \to \infty$ uniformly in t. Pick $\delta > 0$ and define the following intervals for $k \geqslant 1$:

$$I_k := \left[s_0 + \frac{1}{2}\delta(k-1), s_0 + \frac{1}{2}k\delta\right].$$

Then

$$\int_{I_k} e^{2\rho s}\|\bar{V}(s)\|_{L^2}^2\, ds \quad \text{and} \quad \int_{I_k} e^{2\rho s}\|\partial_t\bar{V}(s)\|_{L^2}^2\, ds$$

must converge to zero as $k \to \infty$ in view of

$$\sum_{k=1}^{\infty}\int_{I_k} e^{2\rho s}\|\partial_t\bar{V}(s)\|_{L^2}^2\, ds = \int_{s_0}^{\infty} e^{2\rho s}\|\partial_t\bar{V}(s)\|_{L^2}^2\, ds < \infty$$

and

$$\int_{I_k} e^{2\rho s}\|\bar{V}(s)\|_{L^2}^2\, ds \leqslant T^2\int_{I_k} e^{2\rho s}\|\partial_t\bar{V}(s)\|_{L^2}^2\, ds.$$

Recall that $|\bar{V}(s)|_{C^0([0,T])}$ converges to zero as $s \to \infty$, so $\|\bar{V}(s)\|_{L^2([0,T])}$ depends continuously on s. If we take

$$s_k := \min_{s \in I_k}(e^{2\rho s}\|\partial_t\bar{V}(s)\|_{L^2}^2),$$

then $s_k \to \infty$, $s_{k+1} - s_k \leqslant \delta$, and, in view of (17),

$$\begin{aligned} e^{2\rho s_k}\|\bar{V}(s_k)\|_{L^2}^2 &\leqslant T^2 \cdot e^{2\rho s_k}\|\partial_t\bar{V}(s_k)\|_{L^2}^2 \\ &\leqslant \frac{T^2}{\delta}\int_{I_k} e^{2\rho s}\|\partial_t\bar{V}(s)\|_{L^2}^2\, ds \\ &\to 0 \quad \text{as } k \to \infty. \end{aligned}$$

The Sobolev embedding theorem provides an estimate

$$|\bar{V}(s_k)|_{C^0([0,T])} \leqslant c(\|\bar{V}(s_k)\|_{L^2} + \|\partial_t\bar{V}(s_k)\|_{L^2})$$

with some constant $c > 0$ not depending on k. Hence

$$e^{2\rho s_k}|\bar{V}(s_k)|_{C^0([0,T])} \to 0$$

as $k \to \infty$.

If we split $\bar{V}$ into real part $\bar{B}$ and imaginary part Z, we have

$$e^{\rho s_k}|Z(s_k)|_{C^0([0,T])} \xrightarrow{k\to\infty} 0.$$

Then the differential equation for $\bar{V}$ is

$$\partial_s\bar{B} - \partial_t Z = \bar{H}_1,$$

$$\partial_t\bar{B} + \partial_s Z = \bar{H}_2,$$

$$Z(s,0) \equiv Z(s,T) \equiv 0,$$

which implies $\Delta Z = \partial_s\bar{H}_2 - \partial_t\bar{H}_1 =: \hat{H}$, where $|\hat{H}|$ decays like $\text{const}\cdot e^{-rs}$ as $s \to \infty$.

Define now

$$\varphi(s,t) := e^{\rho s}Z(s,t).$$

Then

$$\begin{aligned}\Delta\varphi(s,t) &= e^{\rho s}\Delta Z(s,t) + 2\rho\partial_s\varphi(s,t) - \rho^2\varphi(s,t)\\ &= e^{\rho s}\hat{H}(s,t) + 2\rho\partial_s\varphi(s,t) - \rho^2\varphi(s,t).\end{aligned}$$

We compute with $\psi(s,t) := |\varphi(s,t)|^2$:

$$\begin{aligned}\Delta\psi(s,t) &= 2|\partial_s\varphi(s,t)|^2 + 2|\partial_t\varphi(s,t)|^2 + 2\langle\varphi(s,t),\Delta\varphi(s,t)\rangle\\ &= 2|\partial_s\varphi(s,t)|^2 + 2|\partial_t\varphi(s,t)|^2 + 2e^{2\rho s}\langle Z(s,t),\hat{H}(s,t)\rangle\\ &\quad + 2\rho\langle\varphi(s,t),\partial_s\varphi(s,t)\rangle - 2\rho^2\psi(s,t).\end{aligned}$$

Hence

$$\begin{aligned}\Delta\psi + 2\rho^2\psi &\geqslant 2|\partial_s\varphi|^2 - 2ce^{(2\rho-r)s} - 2\rho|\varphi||\partial_s\varphi|\\ &\geqslant 2|\partial_s\varphi|^2 - 2ce^{(2\rho-r)s} - \rho^2|\varphi|^2 - |\partial_s\varphi|^2\end{aligned}$$

and

$$\Delta\psi + 3\rho^2\psi \geqslant -2ce^{(2\rho-r)s} =: f(s).$$

Note that $f(s) \to 0$ as $s \to \infty$ since $\rho < r/2$. Recall that $\psi \geqslant 0$, $\psi(s,0) \equiv \psi(s,T) \equiv 0$, and $\psi(s_k,t) \to 0$ uniformly in t for some sequence $s_k \to \infty$ with $s_{k+1} - s_k \leqslant \delta$, where $\delta > 0$ can be chosen arbitrarily small.

Let $\Omega_k := [s_k, s_{k+1}] \times [0,T]$. Our aim is to derive a bound $\psi|_{\Omega_k} \leqslant C$ that does not depend on k, and is therefore valid for the whole strip $[s_0,\infty) \times [0,T]$. Unfortunately, the maximum principle cannot be applied directly because $3\rho^2$ has the wrong sign. However, there are still bounds if Ω_k is sufficiently slim, that is, if δ is chosen sufficiently small.

Define on Ω_k,

$$v_k(s,t) := \sup_{\partial\Omega_k}\psi + (e^{\delta} - e^{s-s_k})\left(3\rho^2\sup_{\Omega_k}\psi + 2ce^{(2\rho-r)s_0}\right) \geqslant 0$$

since $\psi \geqslant 0$ and $s - s_k \leqslant s_{k+1} - s_k \leqslant \delta$. We have

$$\begin{aligned}\Delta v_k(s,t) &= -e^{s-s_k}\left(3\rho^2\sup_{\Omega_k}\psi + 2ce^{(2\rho-r)s_0}\right)\\ &\leqslant -3\rho^2\sup_{\Omega_k}\psi - 2ce^{(2\rho-r)s_0}\end{aligned}$$

and

$$\begin{aligned}\Delta(v_k - \psi) &\leqslant 3\rho^2\left(\psi - \sup_{\Omega_k}\psi\right) + 2c(e^{(2\rho-r)s} - e^{(2\rho-r)s_0})\\ &\leqslant 0.\end{aligned}$$

On the boundary $\partial\Omega_k$, we have

$$\begin{aligned}v_k - \psi &= \sup_{\partial\Omega_k}\psi - \psi|_{\partial\Omega_k} + (e^{\delta} - e^{s-s_k})\left(3\rho^2\sup_{\Omega_k}\psi + 2ce^{(2\rho-r)s_0}\right)\\ &\geqslant 0.\end{aligned}$$

By the weak maximum principle (see [3]), we conclude

$$\inf_{\Omega_k}(v_k - \psi) = \inf_{\partial\Omega_k}(v_k - \psi) \geqslant 0$$

and therefore

$$\psi(s,t) \leqslant \sup_{\partial\Omega_k} \psi + (e^\delta - e^{s-s_k})\left(3\rho^2 \sup_{\Omega_k} \psi + 2ce^{(2\rho-r)s_0}\right),$$

which implies, by taking the supremum on both sides,

$$\sup_{\Omega_k} \psi \leqslant \sup_{\partial\Omega_k} \psi + (e^\delta - 1)\left(3\rho^2 \sup_{\Omega_k} \psi + 2ce^{(2\rho-r)s_0}\right).$$

If we choose now $\delta < \log(1 + 1/3\rho^2)$, then

$$\sup_{\Omega_k} \psi \leqslant \frac{1}{C}\left(\sup_{\partial\Omega_k} \psi + 2c(e^\delta - 1)e^{(2\rho-r)s_0}\right)$$

with $C = 1 - 3\rho^2(e^\delta - 1) > 0$.

In view of the boundary condition on $\psi|_{\partial\Omega_k}$, we see that $\psi|_{\Omega_k}$ is bounded by a constant c that does not depend on k, hence

$$\sup_{(s,t)\in[s_0,\infty)\times[0,T]} e^{\rho s}|Z(s,t)| \leqslant c < \infty.$$

Since the integer l' involved in the definition of $Z = (w_2, \partial_s w_2, \ldots, \partial_s^{l'} w_2)$ was arbitrary, we also have a bound for $e^{\rho s}|\partial_s Z(s,t)|$. Because of $\partial_t \bar{B}(s,t) = \bar{H}_2(s,t) - \partial_s Z(s,t)$, we can estimate

$$|\partial_t \bar{B}(s,t)| \leqslant ce^{-\rho s}.$$

Consider now

$$\bar{B}(s,t) - \bar{B}(s,t') = \int_{t'}^{t} \partial_t \bar{B}(s,\tau)\, d\tau.$$

Integrating over $[0,T]$ with respect to t' and using the fact that $\bar{B}$ has mean value zero, we arrive at

$$\bar{B}(s,t) = \int_0^T \int_{t'}^{t} \partial_t \bar{B}(s,\tau)\, d\tau\, dt',$$

which implies

$$|\bar{B}(s,t)| \leqslant \int_0^T \int_0^T |\partial_t \bar{B}(s,\tau)|\, d\tau\, dt' \leqslant cT^2 e^{-\rho s}.$$

Combining this with the estimate on the mean value of B, we have shown that $|B(s,t)| \leqslant \text{const} \cdot e^{-\rho s}$.

Summarizing, we have shown exponential decay as $s \to \infty$ for $\partial_s^k w$, where $k \geqslant 0$ and $w(s,t) = (b(s,t) - s - b_0, z(s,t) - t)$ with some suitable real constant b_0. Recalling that $\partial_t V = i\,(\partial_s V - H)$, we also obtain exponential decay estimates for $\partial_t V$, which implies estimates for each $\partial_t \partial_s^k w$ $(k \geqslant 0)$. Inductively, we get, finally, exponential decay of $\partial_t^l V$ for each $l \geqslant 0$, which proves Theorem 3.3. □

3.3. A representation formula. We recall from the preceding section that the equation for the finite energy strip

$$\tilde{v} = (b, v) = (b; \zeta, z) : [s_0, \infty) \times [0, T] \to \mathbf{R} \times \mathbf{R}^2 \times \mathbf{R}$$

looks as follows:

- $\partial_s b - \lambda_2(v)\partial_t v = 0$
- $\partial_t b + \lambda_2(v)\partial_s v = 0$
- $\partial_s \zeta + M(\zeta, z)\partial_t \zeta = 0,$

provided s_0 is sufficiently large. Moreover, we have shown that $v(s, \,.\,)$ converges to the characteristic chord $x_0(t) = (0,0,t)$ in $C^\infty([0,T], \mathbf{R}^3)$ as $s \to \infty$, and the convergence is of exponential nature.

The map v satisfies the boundary conditions

$$v(s,0) \in \mathbf{R} \cdot (1,0,0)$$

$$v(s,T) \in \mathbf{R} \cdot (0,1,0) + (0,0,T),$$

and the matrix-valued function M satisfies

$$M^T J_0 M = J_0 \quad \text{with } J_0 = \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix} =: i$$

and

$$-J_0 M > 0.$$

In this section, we want to derive an asymptotic formula for the component ζ of v transversal to the characteristic chord x_0. We define an unbounded linear operator

$$A_\infty : L^2([0,T], \mathbf{R}^2) \supset W_\Gamma^{1,2}([0,T], \mathbf{R}^2) \to L^2([0,T], \mathbf{R}^2)$$

by

$$(A_\infty \cdot \gamma)(t) := -M_\infty(t)\dot{\gamma}(t),$$

where we abbreviate $M_\infty(t) := M(0,0,t)$. Our main result is the following theorem.

THEOREM 3.6. *If ζ does not vanish identically, we have the following asymptotic formula:*

$$\zeta(s,t) = e^{\int_{s_0}^{s} \alpha(\tau)\,d\tau}[e(t) + r(s,t)],$$

where

- *$e \in W_\Gamma^{1,2}([0,T],\mathbf{R}^2)$ is an eigenvector of A_∞ corresponding to some eigenvalue $\lambda < 0$ (here $L^2([0,T],\mathbf{R}^2)$ with the equivalent inner product $(.\,,.) = \int_0^T \langle .\,, -J_0 M_\infty(t).\rangle\,dt$), and A_∞ is selfadjoint;*
- *$\alpha : [s_0,\infty) \to \mathbf{R}$ is a smooth function satisfying $\alpha(s) \to \lambda$ as $s \to \infty$;*
- *$r : [s_0,\infty) \times [0,T] \to \mathbf{R}^2$ is a smooth map with*

$$|\partial^\alpha r(s,t)| \to 0$$

as $s \to \infty$ uniformly in t and $\alpha \in \mathbf{N}^2$ is some multi-index (recall that by convention $0 \in \mathbf{N}$).

The proof of this theorem occupies the rest of this section. We consider again the following inner products on $L^2([0,T],\mathbf{R}^2)$:

$$(u,v)_s := \int_0^T \langle u(t), -J_0 M(\zeta(s,t), z(s,t))v(t)\rangle\,dt.$$

The corresponding norms are denoted by $\|\,.\,\|_s$, while we use the subscript L^2 for the ordinary L^2-norm (or inner product). We saw earlier that these norms are indeed equivalent to the ordinary L^2-norm, and we even have an estimate

$$\frac{1}{C}\|\,.\,\|_{L^2} \leqslant \|\,.\,\|_s \leqslant C\|\,.\,\|_{L^2}$$

with a positive constant C not depending on s. In this section, it is useful to view smooth maps $\zeta : [s_0,\infty) \times [0,T] \to \mathbf{R}^2$ as sections in the trivial vector bundle $E := ([s_0,\infty) \times [0,T]) \times \mathbf{R}^2$ over $[s_0,\infty) \times [0,T]$ and the family $(s,t) \mapsto \langle .\,, -J_0 M(\zeta(s,t), z(s,t))\,.\rangle$ as a bundle metric. In contrast to [7], we cannot use a bundle isomorphism $E \to E$ that transforms $M(v(s,t))$ into $J_0 := \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}$ since this would destroy the boundary condition. So we have to follow a more intrinsic approach. We choose a connection on E by introducing the following covariant derivatives:

$$\nabla_s\zeta(s,t) := \partial_s\zeta(s,t) - \frac{1}{2}M(v(s,t))\partial_s[M(v(s,t))]\cdot\zeta(s,t),$$

$$\nabla_t\zeta(s,t) := \partial_t\zeta(s,t) - \frac{1}{2}M(v(s,t))\partial_t[M(v(s,t))]\cdot\zeta(s,t).$$

We abbreviate

$$\Gamma_1(s,t) := -\frac{1}{2}M(v(s,t))\partial_s[M(v(s,t))]$$

and

$$\Gamma_2(s,t) := -\frac{1}{2}M(v(s,t))\partial_t[M(v(s,t))].$$

For vector fields $X = \alpha_1(\partial/\partial s) + \alpha_2(\partial/\partial t)$, where α_1, α_2 are smooth real-valued functions on $[s_0,\infty)\times[0,T]$, we define $\nabla_X\zeta := \alpha_1\nabla_s\zeta + \alpha_2\nabla_t\zeta$ and observe that

- $\nabla_X(\zeta_1+\zeta_2) = \nabla_X\zeta_1 + \nabla_X\zeta_2$,
- $\nabla_X(f\zeta) = X(f)\cdot\zeta + f\,\nabla_X\zeta$.

We compute for $u_1, u_2 \in W^{1,2}([s_0,\infty)\times[0,T],\mathbf{R}^2)$ with ∇_s being understood in the weak sense:

$$\begin{aligned}\frac{d}{ds}(u_1,u_2)_s &= (\partial_s u_1, u_2)_s + (u_1, \partial_s u_2)_s + \int_0^T \langle u_1(t), -J_0\partial_s[M(v(s,t))]u_2(t)\rangle\,dt \\ &= (u_1,\nabla_s u_2)_s + (\partial_s u_1, u_2)_s + \frac{1}{2}\int_0^T \langle \partial_s[M^T(v(s,t))]J_0u_1(t), u_2(t)\rangle\,dt \\ &= (u_1,\nabla_s u_2)_s + (\partial_s u_1, u_2)_s + \frac{1}{2}\int_0^T \langle -J_0\partial_s[M(v(s,t))]u_1(t), u_2(t)\rangle\,dt \\ &= (u_1,\nabla_s u_2)_s + (\nabla_s u_1, u_2)_s.\end{aligned}$$

If B is a section in the endomorphism bundle $\mathrm{End}(E) = ([s_0,\infty)\times[0,T])\times \mathrm{End}(\mathbf{R}^2)$, then we can define a covariant derivative $\nabla_s B$ by

$$\nabla_s B := \partial_s B + \frac{1}{2}[B\cdot M(v)\partial_s(M(v)) - M(v)\partial_s(M(v))\cdot B],$$

so that $\nabla_s(B\cdot\zeta) = \nabla_s B\cdot\zeta + B\cdot\nabla_s\zeta$. We note that $\nabla_s M(v) = 0$ and $\partial_t\nabla_s\zeta(s,t) - \nabla_s\partial_t\zeta(s,t) = \partial_t\Gamma_1(s,t)\cdot\zeta(s,t)$. We need the following result.

THEOREM 3.7. *Let $T : H \supset D(T) \to H$ be a selfadjoint operator in a Hilbert space H, and let $A_0 : H \to H$ be a linear, bounded, and symmetric operator. Then the following holds:*

- $\mathrm{dist}(\sigma(T), \sigma(T+A_0))$

$$:= \max\left\{\sup_{\lambda\in\sigma(T)} \mathrm{dist}(\lambda,\sigma(T+A_0)),\ \sup_{\lambda\in\sigma(T+A_0)} \mathrm{dist}(\lambda,\sigma(T))\right\}$$

$$\leqslant \|A_0\|_{\mathscr{L}(H)};$$

- *assume further that the resolvent* $(T-\lambda_0)^{-1}$ *of* T *exists and is compact for some* $\lambda_0 \notin \sigma(T)$.

Then $(T-\lambda)^{-1}$ *is compact for every* $\lambda \notin \sigma(T)$, *and* $\sigma(T)$ *consists of isolated eigenvalues* $\{\mu_k\}_{k\in\mathbf{Z}}$ *with finite multiplicities* $\{m_k\}_{k\in\mathbf{Z}}$.

If we assume that $\sup_{k\in\mathbf{Z}} m_k \leqslant M < \infty$ *and that for each* $L > 0$ *there is a number* $m_T(L) \in \mathbf{N}$ *so that every interval* $I \subset \mathbf{R}$ *of length* L *contains at most* $m_T(L)$ *points of* $\sigma(T)$ *(counted with multiplicity), then for each* $L > 0$ *there is also a number* $m_{T+A_0}(L) \in \mathbf{N}$ *so that every interval* $I \subset \mathbf{R}$ *of length* L *contains at most* $m_{T+A_0}(L)$ *points of* $\sigma(T+A_0)$.

Proof. It is sufficient to prove that

$$\sup_{\lambda\in\sigma(T+A_0)} \operatorname{dist}(\lambda,\sigma(T)) \leqslant \|A_0\|_{\mathscr{L}(H)},$$

because $T + A_0$ is also selfadjoint. We pick λ with $\operatorname{dist}(\lambda,\sigma(T)) > \|A_0\|_{\mathscr{L}(H)}$, and we want to show that $\lambda \notin \sigma(T+A_0)$.

We write

$$T + A_0 - \lambda = [\mathrm{Id} + A_0(T-\lambda)^{-1}](T-\lambda).$$

This is invertible with bounded inverse if and only if $(\mathrm{Id} + A_0(T-\lambda)^{-1})^{-1}$ exists and is in $\mathscr{L}(H)$. If $\|A_0(T-\lambda)^{-1}\|_{\mathscr{L}(H)} < 1$, then the Neumann series

$$\mathrm{Id} + \sum_{k=1}^{\infty}(A_0(T-\lambda)^{-1})^k$$

converges in $\mathscr{L}(H)$ to an inverse of $\mathrm{Id} + A_0(T-\lambda)^{-1}$. But

$$\begin{aligned}\|A_0(T-\lambda)^{-1}\|_{\mathscr{L}(H)} &\leqslant \|A_0\|_{\mathscr{L}(H)}\|(T-\lambda)^{-1}\|_{\mathscr{L}(H)}\\ &= \|A_0\|_{\mathscr{L}(H)} \cdot \frac{1}{\operatorname{dist}(\lambda,\sigma(T))}\\ &< 1\end{aligned}$$

since T is selfadjoint, which proves the first part of the theorem. By the spectral mapping theorem, the function

$$\mathbf{R}\backslash\{\lambda_0\} \to \mathbf{R}$$

$$\lambda \mapsto \frac{1}{\lambda-\lambda_0}$$

maps $\sigma(T)$ onto $\sigma((T-\lambda_0)^{-1})$. Since the resolvent $(T-\lambda_0)^{-1}$ is compact, $\sigma((T-\lambda_0)^{-1})$ is a countable set with no accumulation points different from zero. Moreover, each nonzero $\lambda \in \sigma((T-\lambda_0)^{-1})$ is an eigenvalue of $(T-\lambda_0)^{-1}$ with finite multiplicity (see [11, III, 6.7, Theorem 6.29]). It follows that the spectrum of T is a countable set $\{\mu_k\}_{k\in\mathbf{Z}}$ of isolated points and that each $\mu_k \in \sigma(T)$ is an eigenvalue of T with the same multiplicity as $(\mu_k-\lambda_0)^{-1} \in \sigma((T-\lambda_0)^{-1})$. We assume further that all the multiplicities of the eigenvalues μ_k are bounded by some positive constant M. We consider the following family of selfadjoint operators:

$$T(\kappa) := T + \kappa A_0, \quad \kappa \in [-\varepsilon, 1+\varepsilon], \quad \varepsilon > 0.$$

We use the following theorem (for a proof, see [11, VII, 3.5., Theorem 3.9]).

THEOREM 3.8. *Let $T(\kappa) : H \supset D \to H$ be a holomorphic family of selfadjoint operators in a Hilbert space H with domain of definition D so that each $T(\kappa)$ has a compact resolvent ($\kappa \in [-\varepsilon, 1+\varepsilon]$).*

Then there are analytic functions $\mu_n : [0,1] \to \mathbf{R}$ and $\varphi_n : [0,1] \to H$ for each $n \in \mathbf{Z}$ so that $\mu_n(\kappa)$ represent the repeated eigenvalues of $T(\kappa)$ and the $\varphi_n(\kappa)$ form a complete orthonormal family of the associated eigenvectors of $T(\kappa)$.

Remark. The term "holomorphic family" above means that for each $u \in D$, we can expand $T(\kappa)u$ in a convergent Taylor series so that the convergence radius does not depend on u.

We can apply the above theorem since all the operators $T(\kappa)$ have compact resolvent. Using the selfadjointness of $T(\kappa)$ and $(\varphi_n(\kappa), \varphi_n(\kappa)) = 1$, we see that

$$\mu_n(\kappa) = (T(\kappa)\varphi_n(\kappa), \varphi_n(\kappa))$$

and

$$\begin{aligned}\mu_n'(\kappa) &= (A_0\varphi_n(\kappa), \varphi_n(\kappa)) + 2(T(\kappa)\varphi_n(\kappa), \varphi_n'(\kappa)) \\ &= (A_0\varphi_n(\kappa), \varphi_n(\kappa)).\end{aligned}$$

Hence

$$|\mu_n'(\kappa)| \leqslant \|A_0\|_{\mathscr{L}(H)},$$

and therefore

$$|\mu_n(1) - \mu_n(0)| \leqslant \|A_0\|_{\mathscr{L}(H)}.$$

Now let $I = [a,b] \subset \mathbf{R}$ be any interval of length L. Then

$$\begin{aligned} m_{T+A_0}(L) &:= |\{n \in \mathbf{Z} \mid \mu_n(1) \in I\}| \\ &\leqslant |\{n \in \mathbf{Z} \mid \mu_n(0) \in I'\}| \\ &= m_T(L + 2\,\|A_0\|_{\mathscr{L}(H)}), \end{aligned}$$

where $I' = [a - \|A_0\|_{\mathscr{L}(H)}, b + \|A_0\|_{\mathscr{L}(H)}]$. This completes the proof of Theorem 3.7. □

LEMMA 3.9. *If ζ does not vanish identically, then we have for $s \geqslant s_0$,*

$$\|\zeta(s)\|_s = e^{\int_{s_0}^{s} \alpha(\tau)\,d\tau} \|\zeta(s_0)\|_{s_0},$$

where $\alpha : [s_0, \infty) \to \mathbf{R}$ is a smooth function satisfying $\alpha(s) \to \lambda < 0$ as $s \to \infty$ with λ being an eigenvalue of A_∞.

Proof. It is a quite trivial matter to get the required formula. We just write down the correct α. The difficulty is deriving the properties of α as stated in the lemma.

Let us first assume that $\|\zeta(s)\|_s \neq 0$ for all $s \geqslant s_0$. The case where $\|\zeta(s)\|_s = 0$ for some s is treated later (this actually implies $\zeta \equiv 0$). Define

$$\alpha(s) := \frac{(d/ds)\|\zeta(s)\|_s^2}{2\,\|\zeta(s)\|_s^2}.$$

Then $\|\zeta(s)\|_s^2$ satisfies the differential equation

$$\frac{d}{ds}\|\zeta(s)\|_s^2 = 2\alpha(s)\,\|\zeta(s)\|_s^2$$

by definition. But the same differential equation is also solved by

$$F(s) := e^{2\int_{s_0}^{s} \alpha(\tau)\,d\tau} \|\zeta(s_0)\|_{s_0}^2,$$

which implies $F(s) = \|\zeta(s)\|_s^2$.

We introduce now

$$\xi(s,t) := \frac{\zeta(s,t)}{\|\zeta(s)\|_s}.$$

We have

$$\partial_t \xi(s,t) = \frac{\partial_t \zeta(s,t)}{\|\zeta(s)\|_s}$$

and

$$\partial_s \xi(s,t) = \frac{\partial_s \zeta(s,t)}{\|\zeta(s)\|_s} - \frac{\zeta(s,t)}{2\|\zeta(s)\|_s^3} \frac{d}{ds} \|\zeta(s)\|_s^2$$

so that

$$\begin{aligned} 0 &= \nabla_s \xi(s,t) + M(\zeta(s,t), z(s,t))\, \partial_t \xi(s,t) \\ &\quad - \Gamma_1(s,t) \cdot \xi(s,t) + \alpha(s)\, \xi(s,t). \end{aligned} \tag{18}$$

Differentiating $\|\xi(s)\|_s^2 \equiv 1$, we obtain

$$(\nabla_s \xi(s), \xi(s))_s = 0.$$

Taking the L^2-product with $\xi(s)$, we derive the following from equation (18):

$$\alpha(s) = (-M(v(s))\partial_t \xi(s), \xi(s))_s + (\Gamma_1(s) \cdot \xi(s), \xi(s))_s.$$

We conclude that

$$\begin{aligned} \alpha'(s) &= (-\nabla_s[M(v(s))\partial_t \xi(s)], \xi(s))_s + (-M(v(s))\partial_t \xi(s), \nabla_s \xi(s))_s \\ &\quad + (\nabla_s[\Gamma_1(s)\xi(s)], \xi(s))_s + (\Gamma_1(s)\xi(s), \nabla_s \xi(s))_s \\ &= (-M(v(s))\partial_t \nabla_s \xi(s), \xi(s))_s + (M(v(s))\partial_t \Gamma_1(s)\xi(s), \xi(s))_s \\ &\quad + (A(s) \cdot \xi(s), \nabla_s \xi(s))_s + ([\nabla_s \Gamma_1(s)]\xi(s), \xi(s))_s \\ &\quad + 2\,(\Gamma_1(s)\xi(s), \nabla_s \xi(s))_s \\ &= 2\,(A(s) \cdot \xi(s), \nabla_s \xi(s))_s + 2\,(\Gamma_1(s)\xi(s), \nabla_s \xi(s))_s \\ &\quad + (M(v(s))\partial_t \Gamma_1(s)\xi(s), \xi(s))_s + ((\nabla_s \Gamma_1(s))\xi(s), \xi(s))_s. \end{aligned}$$

Inserting

$$A(s) \cdot \xi(s) = \alpha(s)\xi(s) + \nabla_s \xi(s) - \Gamma_1(s)\xi(s),$$

we obtain

$$\alpha'(s) = 2\,\|\nabla_s\xi(s)\|_s^2 + (M(v(s))\partial_t\Gamma_1(s)\xi(s), \xi(s))_s + ([\nabla_s\Gamma_1(s)]\xi(s), \xi(s))_s$$
$$\geqslant 2\,\|\nabla_s\xi(s)\|_s^2 - \varepsilon(s), \tag{19}$$

where $0 < \varepsilon(s) \to 0$ as $s \to \infty$ because $|\partial_t\Gamma_1(s,t)|$ and $|\nabla_s\Gamma_1(s,t)|$ converge to zero uniformly in t for $s \to \infty$.

Assume now that α is not bounded from above. Then we find a sequence $s_k \to \infty$ with $\alpha(s_k) \to \infty$. If we had $\alpha(s) \geqslant \eta$ for some $\eta > 0$ and all large s, then

$$\|\zeta(s)\|_{L^2} \geqslant \frac{1}{C}\|\zeta(s)\|_s \geqslant \frac{1}{C}e^{\eta(s-s_0)}\|\zeta(s_0)\|_{s_0} \xrightarrow{s\to\infty} \infty,$$

which is wrong by the convergence result, Theorem 3.1. Hence for each $\eta > 0$, we can find a sequence $s'_k \to \infty$ with $\alpha(s'_k) < \eta$. Pick now $\eta < \delta$ with δ as in Proposition 3.4, and we may assume that $\alpha(s_k) > \eta$. Now let $\hat{s}_k$ be the smallest number satisfying $\hat{s}_k > s_k$ and $\alpha(\hat{s}_k) = \eta$.

Then η cannot be an eigenvalue of any $A(s)$ since, by Proposition 3.4, $\|A(s)\gamma - \eta\gamma\|_s \geqslant \|A(s)\gamma\|_s - \eta\|\gamma\|_s > 0$ for every $0 \neq \gamma \in W_\Gamma^{1,2}([0,T], \mathbf{R}^2)$. We conclude that

$$\|\nabla_s\xi(\hat{s}_k)\|_{\hat{s}_k} \geqslant \|A(\hat{s}_k)\cdot\xi(\hat{s}_k) - \eta\,\xi(\hat{s}_k)\|_{\hat{s}_k} - \|\Gamma_1(\hat{s}_k)\cdot\xi(\hat{s}_k)\|_{\hat{s}_k}$$
$$\geqslant (\delta - \eta) - C\|\Gamma_1(\hat{s}_k)\cdot\xi(\hat{s}_k)\|_{L^2}$$
$$\geqslant \tau$$

for some $\tau > 0$ since $|\Gamma_1(s,t)| \to 0$ as $s \to \infty$ uniformly in t and $\|\xi(\hat{s}_k)\|_{L^2} \leqslant C$ for all k. Inserting this into the estimate (19), and choosing k so large that $\varepsilon(\hat{s}_k) \leqslant \tau^2$, we obtain

$$\alpha'(\hat{s}_k) \geqslant 2\,\|\nabla_s\xi(\hat{s}_k)\|_{\hat{s}_k}^2 - \varepsilon(\hat{s}_k)$$
$$\geqslant \tau^2 \tag{20}$$
$$> 0.$$

Summarizing, we have

$$\alpha(s_k) > \eta,$$
$$\alpha(\hat{s}_k) = \eta,$$
$$\alpha(s) > \eta \quad \text{for all } s \in [s_k, \hat{s}_k),$$

but (20) implies $\alpha(s) < \eta$ for $s < \hat{s}_k$ close to $\hat{s}_k$, which is a contradiction; hence α is bounded from above.

Let us show now that α is also bounded from below. For this, we need some information about the spectrum of the operators

$$A(s) + \Gamma_1(s) : L^2([0,T],\mathbf{R}^2) \supset W_\Gamma^{1,2}([0,T],\mathbf{R}^2) \to L^2([0,T],\mathbf{R}^2),$$

$$((A(s) + \Gamma_1(s))\,\xi)(t) = -M(v(s,t))\partial_t\xi(s,t) + \Gamma_1(s,t)\cdot\xi(s,t).$$

Let us investigate $A(s)$ first.

Define the matrices

$$T(s,t) := (-J_0M(v(s,t)))^{1/2},$$

$$T_\infty(t) := (-J_0M_\infty(t))^{1/2}.$$

Then T and T_∞ are symmetric and symplectic since this applies to $-J_0M$. Therefore

$$T(s,t)M(v(s,t)) = J_0T(s,t)$$

and, similarly,

$$T_\infty(t)M_\infty(t) = J_0T_\infty(t).$$

A straightforward calculation shows that the maps

$$\Phi_s : (L^2([0,T],\mathbf{R}^2),(\,.\,,\,.\,)_s) \to (L^2([0,T],\mathbf{R}^2),(\,.\,,\,.\,)_{L^2})$$

$$\gamma \mapsto T(s,\,.\,)\gamma,$$

$$\Phi_\infty : (L^2([0,T],\mathbf{R}^2),(\,.\,,\,.\,)) \to (L^2([0,T],\mathbf{R}^2),(\,.\,,\,.\,)_{L^2})$$

$$\gamma \mapsto T_\infty(\,.\,)\gamma$$

are isometries. They map $W_\Gamma^{1,2}([0,T],\mathbf{R}^2)$ onto

$$W_{\Gamma_s}^{1,2}([0,T],\mathbf{R}^2) := \left\{\gamma \in W^{1,2}([0,T],\mathbf{R}^2) \,\middle|\, \begin{matrix} \gamma(0) \in \mathbf{R}\cdot T(s,0)\cdot(1,0) \\ \gamma(T) \in \mathbf{R}\cdot T(s,T)\cdot(0,1) \end{matrix}\right\}$$

and

$$W_{\Gamma_\infty}^{1,2}([0,T],\mathbf{R}^2) := \left\{\gamma \in W^{1,2}([0,T],\mathbf{R}^2) \,\middle|\, \begin{matrix} \gamma(0) \in \mathbf{R}\cdot T_\infty(0)\cdot(1,0) \\ \gamma(T) \in \mathbf{R}\cdot T_\infty(T)\cdot(0,1) \end{matrix}\right\},$$

respectively. We consider the operators

$$\tilde{A}(s) : L^2([0,T],\mathbf{R}^2) \supset W^{1,2}_{\Gamma_s}([0,T],\mathbf{R}^2) \to L^2([0,T],\mathbf{R}^2),$$

$$\tilde{A}(s) := \Phi_s \circ A(s) \circ \Phi_s^{-1},$$

$$\hat{A}_\infty : L^2([0,T],\mathbf{R}^2) \supset W^{1,2}_{\Gamma_\infty}([0,T],\mathbf{R}^2) \to L^2([0,T],\mathbf{R}^2),$$

$$\hat{A}_\infty := \Phi_\infty \circ A_\infty \circ \Phi_\infty^{-1},$$

where we equip $L^2([0,T],\mathbf{R}^2)$ with the ordinary L^2-inner product $(.,.)_{L^2}$. Unitary equivalent selfadjoint operators have the same spectrum; hence

$$\sigma(\tilde{A}(s)) = \sigma(A(s))$$

and

$$\sigma(\hat{A}_\infty) = \sigma(A_\infty).$$

It remains to investigate the spectra of $\tilde{A}(s)$ and $\hat{A}_\infty$. First we note that the operators $\tilde{A}(s)$ and $\hat{A}_\infty$ are selfadjoint. We compute as follows for $\gamma \in W^{1,2}_{\Gamma_s}([0,T],\mathbf{R}^2)$:

$$(\tilde{A}(s)\cdot\gamma)(t) = -J_0\dot{\gamma}(t) + J_0\partial_t T(s,t)T(s,t)^{-1}\gamma(t)$$

$$=: -J_0\dot{\gamma}(t) + S(s,t)\gamma(t).$$

We obtain in the same way, for $\gamma \in W^{1,2}_{\Gamma_\infty}([0,T],\mathbf{R}^2)$,

$$(\hat{A}_\infty\cdot\gamma)(t) = -J_0\dot{\gamma}(t) + J_0\dot{T}_\infty(t)T_\infty(t)^{-1}\gamma(t)$$

$$=: -J_0\dot{\gamma}(t) + S_\infty(t)\gamma(t).$$

Moreover, since T is symmetric,

$$S(s) : L^2([0,T],\mathbf{R}^2) \to L^2([0,T],\mathbf{R}^2)$$

$$(S(s)\cdot\gamma)(t) := S(s,t)\gamma(t)$$

is a linear, bounded, and symmetric operator, which is also the case for S_∞.

Because $v(s,t) \to (0,0,t)$ in $C^\infty([0,T],\mathbf{R}^3)$, we have

$$\|S_\infty - S(s)\|_{\mathscr{L}(L^2([0,T],\mathbf{R}^2))} \to 0$$

as $s \to \infty$. It is still unpleasant that the operators $\hat{A}_\infty$ and $\tilde{A}(s)$ all have different domains of definition. Let us get rid of this problem.

For each s, we choose a smooth path

$$B_s : [0,T] \to \mathrm{SO}(2) = \mathrm{O}(2) \cap \mathrm{Sp}(2)$$

with

- $B_s(0)\ T(s,0)\cdot(1,0) \in \mathbf{R}\cdot T_\infty(0)\cdot(1,0)$,
- $B_s(T)\ T(s,T)\cdot(0,1) \in \mathbf{R}\cdot T_\infty(T)\cdot(0,1)$,
- $B_s(t) \to \mathrm{Id}$ as $s \to \infty$ uniformly in t.

Then we get isometries

$$\Psi_s : L^2([0,T],\mathbf{R}^2) \xrightarrow{\sim} L^2([0,T],\mathbf{R}^2)$$

$$(\Psi_s\gamma)(t) := B_s(t)\gamma(t),$$

mapping $W^{1,2}_{\Gamma_s}([0,T],\mathbf{R}^2)$ onto $W^{1,2}_{\Gamma_\infty}([0,T],\mathbf{R}^2)$. The operators

$$\hat{A}(s) := \Psi_s \circ \tilde{A}(s) \circ \Psi_s^{-1}$$

are also selfadjoint with domain of definition $W^{1,2}_{\Gamma_\infty}([0,T],\mathbf{R}^2)$, and they satisfy $\sigma(\hat{A}(s)) = \sigma(A(s))$. We compute for $\gamma \in W^{1,2}_{\Gamma_\infty}([0,T],\mathbf{R}^2)$:

$$(\hat{A}(s)\cdot\gamma)(t) = (\hat{A}_\infty\cdot\gamma)(t) + \Delta(s,t)\gamma(t),$$

where

$$\Delta(s,t) = [B_s(t)S(s,t)B_s(t)^T - S_\infty(t)] + J_0\dot{B}_s(t)B_s(t)^T$$

is a symmetric perturbation of $\hat{A}_\infty$ with

$$|\Delta(s,t)| \to 0$$

as $s \to \infty$ uniformly in t.

Let us compute now the spectrum of the operator $-J_0(d/dt)$ with domain of definition $W^{1,2}_{\Gamma_\infty}([0,T],\mathbf{R}^2)$. Identifying $\mathbf{R}^2$ with $\mathbf{C}$, we write

$$\mathbf{R}\cdot T_\infty(0)\cdot(1,0) = \mathbf{R}e^{i\varphi_1},$$

$$\mathbf{R}\cdot T_\infty(T)\cdot(0,1) = \mathbf{R}e^{i\varphi_2}.$$

Then we have to consider the operator $-i(d/dt)$ acting on paths $\gamma\colon [0,T] \to \mathbf{C}$ that satisfy the boundary condition

$$\gamma(0) \in \mathbf{R}e^{i\varphi_1}$$

$$\gamma(T) \in \mathbf{R}e^{i\varphi_2}.$$

The eigenvalues of $-i(d/dt)$ are then given by

$$\mu_k = \frac{\varphi_1 - \varphi_2}{T} + \frac{k\pi}{T}$$

with $k \in \mathbf{Z}$, and the corresponding eigenfunctions are

$$\gamma_k(t) = a \cdot e^{i(\mu_k t + \varphi_1)}, \quad a \in \mathbf{R};$$

and so the eigenvalues all have multiplicity 1. Moreover, the resolvent is compact.

So we have also shown that the spectrum of $-J_0(d/dt)$ with domain of definition $W^{1,2}_{\Gamma_\infty}([0,T],\mathbf{R}^2)$ consists only of the eigenvalues $\{\mu_k\}_{k\in\mathbf{Z}}$ that all have multiplicity 1, and the distance of two neighbouring eigenvalues equals π/T. Then by Theorem 3.7, we find for all $L > 0$ some $m \in \mathbf{N}$ so that every interval $I \subseteq \mathbf{R}$ of length L contains at most m points of the spectrum of A_∞. Moreover,

$$\operatorname{dist}(\sigma(A_\infty), \sigma(A(s))) \to 0 \tag{21}$$

as $s \to \infty$.

Define now the intervals

$$I_n := [-(n+1)L, -nL], \quad n \in \mathbf{N}.$$

Then each I_n contains at most m points of $\sigma(A_\infty)$, so there is a closed subinterval $J_n \subset I_n$ of length $L/(m+1)$ that does not contain any point of $\sigma(A_\infty)$. Because of (21), there is a closed interval $J_n' \subseteq J_n \subseteq I_n$ of length $L/2(m+1)$ that does not contain any point of $\sigma(A(s))$ whenever $s \geqslant s_1$ where s_1 is sufficiently large (this s_1 does not depend on n).

So we found a sequence $r_n \to -\infty$ and a positive constant d', so that

$$[r_n - d', r_n + d'] \cap \sigma(A(s)) = \emptyset$$

for all large s. Replacing d' by a smaller constant d, we still have

$$[r_n - d, r_n + d] \cap \sigma(A(s) + \Gamma_1(s)) = \emptyset$$

for all large s.

Recall the differential equation for ξ:

$$(A(s) + \Gamma_1(s)) \cdot \xi(s) - \alpha(s) \cdot \xi(s) = \nabla_s \xi(s).$$

If α were not to be bounded from below, then we could find a sequence $s_n \to \infty$ with $\alpha(s_n) = r_n$ and $\alpha'(s_n) < 0$. Since $A + \Gamma_1$ is selfadjoint, we have the following for any θ in the resolvent set:

$$\|(A(s) + \Gamma_1(s) - \theta \cdot \mathrm{Id})^{-1}\|_s = \frac{1}{\mathrm{dist}(\theta, \sigma(A(s) + \gamma_1(s)))}.$$

We estimate

$$\begin{aligned} 1 = \|\xi(s_n)\|_{s_n} &= \|(A(s_n) + \Gamma_1(s_n) - r_n \cdot \mathrm{Id})^{-1} \nabla_s \xi(s_n)\|_{s_n} \\ &\leqslant \mathrm{dist}(r_n, \sigma(A(s_n) + \Gamma_1(s_n)))^{-1} \|\nabla_s \xi(s_n)\|_{s_n} \\ &\leqslant \frac{1}{d} \|\nabla_s \xi(s_n)\|_{s_n}, \end{aligned}$$

and therefore

$$\|\nabla_s \xi(s_n)\|_{L^2} \geqslant \tau$$

for some suitable $\tau > 0$ and all $n \in \mathbf{N}$. Using the estimate (18) for $\alpha'(s)$, we conclude that

$$\alpha'(s_n) \geqslant \tau^2 > 0$$

if n is large enough. This is in contradiction to our assumption, so α is also bounded from below.

There exists a sequence $s_k \to \infty$ so that $\|\nabla_s \xi(s_k)\|_{s_k} \to 0$. Otherwise, we have for all large s,

$$\|\nabla_s \xi(s)\|_s \geqslant \eta > 0$$

with some suitable η. But then because of (18),

$$\alpha'(s) \geqslant \eta^2 > 0$$

for $s \geqslant s_1$ (s_1 sufficiently large) and

$$\alpha(s) \geqslant \eta^2(s - s_1) + \alpha(s_1)$$

so that $\|\zeta(s)\|_s \to \infty$ as $s \to \infty$, which is not true.

Because α is bounded, we can find a subsequence (which we also denote by $(s_k)_{k\in\mathbf{N}}$) so that

$$\lim_{k\to\infty}\alpha(s_k)=\lambda$$

exists. We claim that $\lambda\in\sigma(A_\infty)$. If we had $\lambda\notin\sigma(A_\infty)$, then $\varepsilon:=\inf_{\mu\in\sigma(A_\infty)}|\lambda-\mu|>0$ because $\sigma(A_\infty)$ is closed, and therefore

$$|\mu'-\lambda|\geqslant\varepsilon-|\mu-\mu'|\quad\forall\mu\in\sigma(A_\infty),\ \mu'\in\sigma(A(s)+\Gamma_1(s)).$$

This implies

$$\operatorname{dist}(\lambda,\sigma(A(s)+\Gamma_1(s)))\geqslant\varepsilon-\sup_{\mu'\in\sigma(A(s))}\operatorname{dist}(\mu',\sigma(A_\infty))>\varepsilon/2$$

if s is sufficiently large, by Theorem 3.7; that is,

$$\alpha(s_k)\notin\sigma(A(s_k)+\Gamma_1(s_k))$$

for k sufficiently large.

Then

$$\begin{aligned}\frac{4}{\varepsilon}\|\nabla_s\xi(s_k)\|_{s_k}&>\operatorname{dist}(\alpha(s_k),\sigma(A(s_k)+\Gamma_1(s_k)))^{-1}\|\nabla_s\xi(s_k)\|_{s_k}\\&\geqslant\|(A(s_k)+\Gamma_1(s_k)-\alpha(s_k)\mathrm{Id})^{-1}\nabla_s\xi(s_k)\|_{s_k}\\&=\|\xi(s_k)\|_{s_k}\\&=1,\end{aligned}$$

where k is chosen so large that $|\lambda-\alpha(s_k)|<\varepsilon/4$. But this contradicts $\|\nabla_s\xi(s_k)\|_{s_k}\to 0$, hence $\lambda\in\sigma(A_\infty)$. Let us show that indeed

$$\lim_{s\to\infty}\alpha(s)=\lambda.$$

Take now an arbitrary sequence $s'_k\to\infty$. Then there is a subsequence that we denote again by (s'_k) so that $\alpha(s'_k)$ converges to some μ, and we have to show that $\mu=\lambda$. Assume that $\mu<\lambda$. It is a consequence of Theorem 3.7 that there are $d>0$ and $\nu\in(\mu,\lambda)$ so that

$$\nu\notin\sigma(A(s)+\Gamma_1(s))$$

and

$$\operatorname{dist}(\nu, \sigma(A(s) + \Gamma_1(s))) \geqslant d$$

whenever s is sufficiently large.

Let $\hat{s}$ be any number with $\alpha(\hat{s}) = \nu$. Then we estimate as before

$$\frac{1}{d}\|\nabla_s \xi(\hat{s})\|_{\hat{s}} \geqslant \operatorname{dist}(\nu, \sigma(A(\hat{s}) + \Gamma_1(\hat{s})))^{-1}\|\nabla_s \xi(\hat{s})\|_{\hat{s}} \geqslant 1$$

and

$$\alpha'(\hat{s}) \geqslant d^2 > 0,$$

where $\hat{s}$ is large, which implies $\alpha(s) > \nu$ for s sufficiently large in contradiction to $\alpha(s_k') \to \mu < \nu$.

In the case $\mu > \lambda$, we also get $\alpha'(\hat{s}) \geqslant d^2 > 0$ for all large $\hat{s}$ satisfying $\alpha(\hat{s}) = \nu \in (\lambda, \mu)$ (ν, d having the same properties as before), which would imply $\alpha(s) > \nu$ in contradiction to $\alpha(s_k) \to \lambda < \nu$.

Hence we have shown that $\alpha(s)$ converges to $\lambda \in \sigma(A_\infty)$ as $s \to \infty$. The number λ is actually an eigenvalue because $(A_\infty - \mu)^{-1}$ is a compact operator whenever $\mu \notin \sigma(A_\infty)$. We must have $\lambda \leqslant 0$ since otherwise we would have $\|\zeta(s)\|_{L^2} \to \infty$ as $s \to \infty$. Because the spectrum of A_∞ consists of eigenvalues only and in view of the nondegeneracy assumption we have $0 \notin \sigma(A_\infty)$, and therefore $\lambda < 0$.

So we have settled the case for which $\|\zeta(s)\|_{L^2} \neq 0$ for all $s \geqslant s_0$. Assume now that $\|\zeta(s^*)\|_{L^2} = 0$ for some $s^* \geqslant s_0$. Then $\zeta(s^*, t^*) = 0$ for all $t^* \in [0, T]$. Using the generalized similarity principle (see [10] or [1]), we find an open neighbourhood U for each (s^*, t^*) so that $\zeta|_U$ is represented by

$$\zeta(s,t) = \Phi(s,t)h(s,t),$$

where $\Phi : U \to GL_{\mathbf{R}}(\mathbf{C})$ is continuous and $h : U \to \mathbf{C}$ is a holomorphic function. Because (s^*, t^*) is a cluster point of zeroes, we conclude that $\zeta|_U \equiv 0$ and consequently $\zeta(s,t) \equiv 0$ for all $s \geqslant s_0$. This finally completes the proof of Lemma 3.9. □

LEMMA 3.10. *For every $\beta = (\beta_1, \beta_2) \in \mathbf{N}^2$ and $j \in \mathbf{N}$, we have*

$$\sup_{(s,t) \in [s_0, \infty) \times [0,T]} \left|\partial^\beta \xi(s,t)\right| < \infty,$$

$$\sup_{s_0 \leqslant s < \infty} \left|\frac{d^j \alpha}{ds^j}(s)\right| < \infty,$$

where $\xi(s,t) = \zeta(s,t)/\|\zeta(s)\|_s$ *and* $\alpha(s) = (A(s)\cdot\xi(s) + \Gamma_1(s)\cdot\xi(s), \xi(s))_s$ *(recall that* $0 \in \mathbf{N}$ *by convention).*

Proof. Recall that ξ solves the following equation:

$$\partial_s\xi(s,t) = -M(v(s,t))\partial_t\xi(s,t) - \alpha(s)\cdot\xi(s,t). \tag{22}$$

Moreover, for any $k \geqslant 0$,

$$\partial_s^k\xi(s,0) \in \mathbf{R},$$

$$\partial_s^k\xi(s,T) \in i\mathbf{R}.$$

We show the following. For each $N \in \mathbf{N}^*(N \geqslant 1!), 2 < p < \infty$ there are $\delta_N, C_{N,p} > 0$ and $s_N^* \geqslant s_0$ so that

$$\|\xi\|_{W^{N,p}([s^*-\delta_N, s^*+\delta_N]\times[0,T],\mathbf{R}^2)} \leqslant C_{N,p}$$

whenever $s^* \geqslant s_N^*$. The constant $C_{N,p}$ does not depend on s^*. By the Sobolev embedding theorem, we obtain

$$\|\xi\|_{C^{N-1}(Q_{\delta_N})} \leqslant \tilde{C}_{N,p},$$

where $Q_{\delta_N} := Q(s^*,\delta_N) := [s^*-\delta_N, s^*+\delta_N]\times[0,T]$ and $\tilde{C}_{N,p} > 0$ does not depend on $s^* \geqslant s_N^*$. But then

$$\sup_{(s,t)\in[s_0,\infty)\times[0,T]} |\partial^\beta\xi(s,t)| \leqslant \max\left\{\sup_{(s,t)\in[s_0,s_N^*]\times[0,T]} |\partial^\beta\xi(s,t)|, \tilde{C}_{N,p}\right\} < \infty$$

if $|\beta| \leqslant N-1$. Take $\delta_0 > 0$ and define a sequence $\delta_j \searrow (1/2)\delta_0$ by $\delta_j := (1/2)\delta_0(1+2^{-j})$. Let $\beta_j : \mathbf{R} \to [0,1]$ be a smooth function that vanishes outside $(s^*-\delta_{j-1}, s^*+\delta_{j-1})$ and equals 1 on $[s^*-\delta_j, s^*+\delta_j]$.

We define the column vector

$$W(s,t) := (\xi(s,t), \partial_s\xi(s,t), \ldots, \partial_s^{N-1}\xi(s,t)).$$

If $F : [s_0,\infty)\times[0,T] \to \mathbf{C}^N$ is a smooth function with $F(s,0) \in V_1, F(s,T) \in V_2$, where V_1, V_2 are totally real subspaces of $\mathbf{C}^N$, then we have the following a priori estimate for $\beta_j F$:

$$\|\beta_j F\|_{W^{1,p}(Q(s^*,\delta_{j-1}))} \leqslant C\|\bar{\partial}(\beta_j F)\|_{L^p(Q(s^*,\delta_{j-1}))}, \tag{23}$$

where C is a positive constant depending on p, δ_{j-1}, but not on s^*, and where $\bar{\partial} = \partial_s + J_0\partial_t$ is the standard Cauchy-Riemann operator. (For a proof of this estimate see [1], [12] or [13].)

We look for a differential equation that is satisfied by W and derive an a priori estimate using (23). We apply now ∂_s^k to equation (22) with $k \geqslant 1$ and obtain

$$0 = \partial_s(\partial_s^k \xi) + M(v)\partial_t(\partial_s^k \xi) + \sum_{l=1}^{k} \binom{k}{l} \partial_s^l(M(v))\partial_t(\partial_s^{k-l}\xi) + \sum_{l=0}^{k} \binom{k}{l} \frac{d^l\alpha}{ds^l} \cdot \partial_s^{k-l}\xi.$$

As in the proof of Theorem 3.3, we introduce

$$\hat{\Delta} := \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ \Delta_{11} & 0 & 0 & \cdots & 0 & 0 \\ \Delta_{22} & \Delta_{12} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \Delta_{N-1,N-1} & \Delta_{N-2,N-1} & \Delta_{N-3,N-1} & \cdots & \Delta_{1,N-1} & 0 \end{pmatrix}$$

with $\Delta_{lk} := \binom{k}{l}\partial_s^l(M(v))$ and

$$\hat{\alpha} := \begin{pmatrix} \alpha & 0 & 0 & \cdots & 0 \\ \alpha_{11} & \alpha & 0 & \cdots & 0 \\ \alpha_{22} & \alpha_{12} & \alpha & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \alpha_{N-1,N-1} & \alpha_{N-2,N-1} & \alpha_{N-3,N-1} & \cdots & \alpha \end{pmatrix}$$

with $\alpha_{lk} := \binom{k}{l}\, d^l\alpha/ds^l$. Then we obtain the following equation for W:

$$\partial_s W + M(v)\partial_t W + \hat{\Delta} \cdot \partial_t W + \hat{\alpha} \cdot W = 0, \tag{24}$$

where $|\hat{\Delta}(s,t)| \to 0$ uniformly in t as $s \to \infty$.

(Remark on the notation: We write again $M(v)$, J_0, M_∞ for $\operatorname{diag}(M(v), \ldots, M(v))$, $\operatorname{diag}(J_0, \ldots, J_0)$, and $\operatorname{diag}(M_\infty, \ldots, M_\infty)$.)

We have to bring the Cauchy-Riemann operator $\bar{\partial}$ into play as follows. Define

$$T_\infty(t) := (-J_0 M_\infty(t))^{1/2}$$

and recall that $T_\infty(t) M_\infty(t) T_\infty(t)^{-1} = J_0$. Applying T_∞ to equation (24), we obtain

$$
\begin{aligned}
0 &= \partial_s(T_\infty W) + T_\infty M_\infty T_\infty^{-1} T_\infty \partial_t W + T_\infty \hat{\Delta} \partial_t W \\
&\qquad + T_\infty(\hat{\alpha} \cdot W) + T_\infty(M(v) - M_\infty)\partial_t W \\
&= \bar{\partial}(T_\infty W) - J_0 \dot{T}_\infty W + T_\infty(\hat{\alpha} \cdot W) + T_\infty \hat{\Delta} \partial_t W + T_\infty(M(v) - M_\infty)\partial_t W \\
&=: \bar{\partial}(T_\infty W) + \Delta_1 \cdot \partial_t W - J_0 \dot{T}_\infty W + T_\infty(\hat{\alpha} \cdot W),
\end{aligned}
$$

where again $|\Delta_1(s,t)| \to 0$ as $s \to \infty$ uniformly in t.

We estimate with (23), $\varepsilon_j(s^*) = \sup_{Q(s^*,\delta_{j-1})}|\Delta_1|$ and $(\beta_j W)(s,t) := \beta_j(s)W(s,t)$:

$$
\begin{aligned}
\|\beta_j W\|_{W^{1,p}(Q_{\delta_{j-1}})} &\leqslant c_1 \|\beta_j(T_\infty W)\|_{W^{1,p}(Q_{\delta_{j-1}})} \\
&\leqslant c_2 \|\bar{\partial}(\beta_j(T_\infty W))\|_{L^p(Q_{\delta_{j-1}})} \\
&\leqslant c_2 \|\beta_j'\|_{C^0} \|W\|_{L^p(Q_{\delta_{j-1}})} + c_2 \|\beta_j \bar{\partial}(T_\infty W)\|_{L^p(Q_{\delta_{j-1}})} \\
&\leqslant (c_2 \|\beta_j'\|_{C^0} + c_3)\|W\|_{L^p(Q_{\delta_{j-1}})} + c_4 \|\hat{\alpha} \cdot W\|_{L^p(Q_{\delta_{j-1}})} \\
&\quad + \varepsilon_j(s^*)\|\partial_t(\beta_j W)\|_{L^p(Q_{\delta_{j-1}})} \\
&\leqslant c_5 \|W\|_{L^p(Q_{\delta_{j-1}})} + c_4 \|\hat{\alpha} \cdot W\|_{L^p(Q_{\delta_{j-1}})} + \varepsilon_j(s^*)\|\beta_j W\|_{W^{1,p}(Q_{\delta_{j-1}})}.
\end{aligned}
$$

Remarks. (i) Recalling that T_∞ and M_∞ are actually $\mathrm{diag}(T_\infty, \ldots, T_\infty)$ and $\mathrm{diag}(M_\infty, \ldots, M_\infty)$, we conclude that $T_\infty(0)\mathbf{R}^N$ and $T_\infty(T)J_0\mathbf{R}^N$, are totally real subspaces of $\mathbf{C}^N$ so that the estimate (23) can be applied to $T_\infty W$.

(ii) The constants $c_1, \ldots, c_5$ above do not depend on s^* since they only contain the constant C in (23) or upper bounds for T_∞, T_∞^{-1} and their derivatives.

Choose now $s_j^* \geqslant s_0$ so that $\varepsilon_j(s^*) \leqslant 1/2$ whenever $s^* \geqslant s_j^*$; hence

$$
\frac{1}{2}\|\beta_j W\|_{W^{1,p}(Q_{\delta_{j-1}})} \leqslant c_5 \|W\|_{L^p(Q_{\delta_{j-1}})} + c_4 \|\hat{\alpha} \cdot W\|_{L^p(Q_{\delta_{j-1}})}
$$

and

$$
\|W\|_{W^{1,p}(Q_{\delta_j})} \leqslant c_0(\|W\|_{L^p(Q_{\delta_{j-1}})} + \|\hat{\alpha} \cdot W\|_{L^p(Q_{\delta_{j-1}})}) \tag{25}
$$

for a suitable constant $c_0 = c_0(N, p, j)$ independent from s^* whenever $s^* \geqslant s_j^*$.

We now proceed inductively. We first discuss the case where $N = 1$; that is, $W = (\xi)$, $\hat{\alpha} = (\alpha)$, and $\hat{\Delta}$ vanishes. We know already that α is bounded and

$$\|W\|^2_{L^2(Q_{\delta_0})} = \int_{s^*-\delta_0}^{s^*+\delta_0} \|\xi(s)\|^2_{L^2([0,T])}\, ds \leqslant 2C^2\delta_0$$

since $\|\xi(s)\|_s = 1$.

Using (25), we get for $s^* \geqslant s_1^*$,

$$\|W\|_{W^{1,2}(Q_{\delta_1})} \leqslant c \leqslant \infty,$$

where c depends on δ_0 but not on s^*. By the Sobolev embedding theorem, we also have bounds

$$\|W\|_{L^p(Q_{\delta_1})} \leqslant c_p < \infty$$

not depending on s^*. Inserting this into (25) again, we obtain

$$\|W\|_{W^{1,p}(Q_{\delta_2})} \leqslant c_p' < \infty \tag{26}$$

for all $Q_{\delta_2} = Q(s^*, \delta_2)$ with $s^* \geqslant s_2^*$. We have shown before (equation (19)) that

$$\alpha'(s) = 2\|\nabla_s\xi(s)\|_s^2 + (M(v(s))\partial_t\Gamma_1(s)\xi(s), \xi(s))_s + ([\nabla_s\Gamma_1(s)]\xi(s), \xi(s))_s,$$

which implies

$$|\alpha'(s)| \leqslant c'\|\partial_s\xi(s)\|^2_{L^2} + c''$$

for suitable constants $c', c'' > 0$. Therefore,

$$\begin{aligned}
\|\alpha'\|^p_{L^p(s^*-\delta_2, s^*+\delta_2)} &\leqslant \int_{s^*-\delta_2}^{s^*+\delta_2} (c'\|\partial_s\xi(s)\|^2_{L^2} + c'')^p\, ds \\
&\leqslant 2^{p-1}\int_{s^*-\delta_2}^{s^*+\delta_2} ((c')^p\|\partial_s\xi(s)\|^{2p}_{L^2} + (c'')^p)\, ds \\
&= (c')^p 2^{p-1}\int_{s^*-\delta_2}^{s^*+\delta_2} \left(\int_0^T |\partial_s\xi(s,t)|^2\, dt\right)^p ds + 2^p\delta_2(c'')^p \\
&\leqslant (c')^p 2^{p-1} T^{p-1}\|\partial_s\xi\|^{2p}_{L^{2p}(Q_{\delta_2})} + 2^p\delta_2(c'')^p \\
&\leqslant c''',
\end{aligned}$$

where $0 < c'''$ does not depend on s^*, using Hölder's inequality and (26) with p replaced by $2p$.

Assume now that $W = (\xi, \partial_s\xi, \ldots, \partial_s^{N-1}\xi)$ is bounded in $L^p(Q(s^*,\delta_N))$ by a constant $c = c(N,p) > 0$ that does not depend on s^* (s^* larger than some s_N^*). Moreover, we assume that all the derivatives of α up to order $N-1$ are bounded in $L^p(s^* - \delta_N, s^* + \delta_N)$ by constants not depending on s^*.

Then by (25),

$$\|W\|_{W^{1,p}(Q_{\delta_{N+1}})} \leqslant c_0(\|W\|_{L^p(Q_{\delta_N})} + \|\hat{\alpha} \cdot W\|_{L^p(Q_{\delta_N})})$$

with $0 < c_0 = c_0(N,p)$ whenever $s^* \geqslant s_{N+1}^*$. Each component of $\hat{\alpha} \cdot W$ can be written as

$$\sum_{l=0}^{N-1} k_l \frac{d^l\alpha}{ds^l}(s)\partial_s^{N-1-l}\xi(s,t)$$

with suitable real constants k_l.

Now $\xi, \partial_s\xi, \ldots, \partial_s^{N-2}\xi$ are bounded in $W^{1,p}(Q_{\delta_N})$, so by the Sobolev embedding theorem, we have $C^0(Q_{\delta_N})$-bounds independent of $s^* \geqslant s_N^*$. The derivative $\partial_s^{N-1}\xi$ is only (s^*-uniformly) bounded in L^p, but it is paired with α, which we know to be bounded; hence $\hat{\alpha} \cdot W$ is bounded in $L^p(Q(s^*,\delta_N))$ independent of s^*. This gives us an s^*-uniform $W^{1,p}(Q_{\delta_{N+1}})$-bound on W.

In the next induction step (i.e., $W = (\xi, \partial_s\xi, \ldots, \partial_s^N\xi)$), the Nth derivative of α appears in $\hat{\alpha} \cdot W$, so we still have to show that $d^N\alpha/ds^N$ is (s^*-uniformly) bounded in $L^p(s^* - \delta_{N+1}, s^* + \delta_{N+1})$.

Using (19), we note that $d^N\alpha/ds^N$ can be expressed as follows:

$$\begin{aligned}\frac{d^N\alpha}{ds^N} = &\sum_{k_1+k_2=N-1} c_{k_1,k_2}(\nabla_s^{k_1+1}\xi(s), \nabla_s^{k_2+1}\xi(s))_s \\ &+ \sum_{l_1+l_2+l_3=N-1} d_{l_1,l_2,l_3}([\nabla_s^{l_1+1}\Gamma_1(s)]\nabla_s^{l_2}\xi(s), \nabla_s^{l_3}\xi(s))_s \\ &+ \sum_{m_1+m_2+m_3=N-1} e_{m_1,m_2,m_3}(M(v(s))\nabla_s^{m_1}\partial_t\Gamma_1(s)\nabla_s^{m_2}\xi(s), \nabla_s^{m_3}\xi(s))_s, \qquad (27)\end{aligned}$$

where the indices k_j, l_j, m_j range from zero to $N-1$, and c, d, e are suitable real constants. The s^*-uniform $W^{1,p}(Q_{\delta_{N+1}})$-bound on $W = (\xi, \partial_s\xi, \ldots, \partial_s^{N-1}\xi)$ implies an s^*-uniform C^0-bound by the Sobolev embedding theorem. Since $\Gamma_1(s)$ and all its derivatives converge to zero as $s \to \infty$ uniformly in t, we also obtain s^*-uniform C^0-bounds for all covariant derivatives $\nabla_x^m\xi(s)$ up to order $m = N-1$, and s^*-uniform L^p-bounds for $\nabla_s^N\xi(s)$. The only expression in (27) that contains $\nabla_s^N\xi(s)$ is $(\nabla_s^N\xi(s), \xi(s))_s$, so $\|d^N\alpha/ds^N\|_{L^p(Q_{\delta_{N+1}})}$ can be estimated from above by the C^0-bounds on $\nabla_s^m\xi(s)$ for $m \leqslant N-1$ and the L^{2p}-bound on $\nabla_s^N\xi(s)$.

Summarizing, we have shown the following. For each $N \in \mathbf{N}^*$ and $2 < p < \infty$, there are constants $c = c(N,p) > 0$ and $s_N^* \geqslant s_0$ so that

$$\left\|\frac{d^k\alpha}{ds^k}\right\|_{L^p(s^*-\delta_N, s^*+\delta_N)}, \ \|\partial_s^k\partial_t^l\xi\|_{L^p(Q(s^*,\delta_N))} \leqslant c(N,p)$$

whenever $s^* \geqslant s_N^*$, $0 \leqslant k \leqslant N-1$, and $l \in \{0,1\}$. Applying the Sobolev embedding theorem to α, we get the required bounds

$$\sup_{s_0 \leqslant s < \infty}\left|\frac{d\alpha}{ds}(s)\right| < \infty.$$

In order to complete the proof, we still have to derive s^*-uniform $L^p(Q(s^*,\delta_N))$-bounds for $\partial_s^k\partial_t^l\xi$ when $l > 1$. We do this by applying $\partial_s^k\partial_t^{l-1}$ to

$$\partial_t\xi(s,t) = M(v(s,t))\partial_s\xi(s,t) + \alpha(s)M(v(s,t)) \cdot \xi(s,t).$$

Then one can proceed inductively, and the proof of Lemma 3.10 is completed. □

The following lemma is proved in the same way as Lemma 3.6. in [7]. We only have to use the covariant derivative $\nabla_s\xi$ in the estimates instead of $\partial_s\xi$. So we omit the proof.

LEMMA 3.11. *Let*

$$E \subseteq W_\Gamma^{1,2}([0,T],\mathbf{R}^2) \subseteq L^2([0,T],\mathbf{R}^2)$$

be the eigenspace of A_∞ belonging to $\lambda \in \sigma(A_\infty)$. Then

$$\inf_{e \in E}\|\xi(s) - e\|_{W^{1,2}([0,T],\mathbf{R}^2)} \to 0$$

as $s \to \infty$. □

LEMMA 3.12. *There exists $e \in E$ such that $\xi(s) \to e$ in $W^{1,2}([0,T],\mathbf{R}^2)$ as $s \to \infty$.*

Proof. Take any sequence $s_n \to \infty$. By Lemma 3.10, $(\xi(s_n))_{n\in\mathbf{N}}$ is bounded in $W^{2,2}([0,T],\mathbf{R}^2)$. Since $W^{2,2}([0,T],\mathbf{R}^2)$ is compactly embedded in $W^{1,2}([0,T],\mathbf{R}^2)$, we find a subsequence of (s_n) (which we denote again by (s_n)) that converges in $W^{1,2}([0,T],\mathbf{R}^2)$ to some e. Using Lemma 3.11, we conclude that $e \in E$. So every sequence $s_n \to \infty$ has a subsequence (s_n') so that $\xi(s_n')$ converges in $W^{1,2}([0,T],\mathbf{R}^2)$ to some eigenvector of A_∞.

It remains to show that this limit is unique. So assume that

- $\xi(\tau_n) \to e' \in E$,
- $\xi(s_n) \to e \in E$

in $W^{1,2}([0,T],\mathbf{R}^2)$ for sequences $s_n, \tau_n \to \infty$, and show that $e = e'$. We equip $L^2([0,T],\mathbf{R}^2)$ with the inner product

$$(u_1, u_2) := \int_0^T \langle u_1(t), -J_0 M_\infty(t) u_2(t) \rangle \, dt,$$

and we denote the corresponding norm by $\|\,.\,\|$. Let

$$P : L^2([0,T],\mathbf{R}^2) \to E$$

be the orthogonal projection onto the eigenspace E of A_∞ belonging to the eigenvalue λ, and let

$$\hat{\xi}(s,t) := (P\xi(s))(t).$$

We claim that $A_\infty(P\xi(s)) = P(A_\infty \xi(s))$. Indeed

$$\begin{aligned} A_\infty \cdot \xi(s) &= A_\infty((\mathrm{Id} - P)\xi(s)) + A_\infty(P\xi(s)) \\ &= A_\infty((\mathrm{Id} - P)\xi(s)) + \lambda\,(P\xi(s)) \end{aligned}$$

and

$$P(A_\infty \cdot \xi(s)) = P(A_\infty((\mathrm{Id} - P)\xi(s))) + A_\infty(P\xi(s)).$$

If $\kappa \in E$, then

$$(A_\infty \cdot \xi(s) - A_\infty(P\xi(s)), \kappa) = (\xi(s) - P\xi(s), \lambda \cdot \kappa) = 0,$$

and therefore $P(A_\infty((\mathrm{Id} - P)\xi(s))) = 0$, proving the claim. Recall that

$$\partial_s \xi(s,t) = (A(s) \cdot \xi(s))(t) - \alpha(s)\xi(s,t).$$

Applying P, we obtain with $\varepsilon(s) := A(s) - A_\infty$,

$$\begin{aligned} \partial_s \hat{\xi}(s,t) &= (A_\infty \cdot \hat{\xi}(s))(t) - \alpha(s)\hat{\xi}(s,t) + (P\varepsilon(s)\xi(s))(t) \\ &= (\lambda - \alpha(s))\hat{\xi}(s,t) + (P\varepsilon(s)\xi(s))(t) \end{aligned} \tag{28}$$

and

$$
\begin{aligned}
&\|P((A(s)-A_\infty)\xi(s))\| \\
&\quad \leqslant \|(A(s)-A_\infty)\xi(s)\| \\
&\quad \leqslant \left\| \left[\int_0^1 DM(\sigma v(s,t)+(0,0,(1-\sigma)t))\cdot(v(s,t)-(0,0,t))\,d\sigma \right] \cdot \partial_t \xi(s,t) \right\| \\
&\quad \leqslant c\, e^{-\rho s},
\end{aligned}
$$

using Lemma 3.10 and Theorem 3.3. Define

$$
\eta(s,t) := \frac{\hat{\xi}(s,t)}{\|\hat{\xi}(s)\|}.
$$

Then we obtain with (28)

$$
\begin{aligned}
\partial_s \eta(s,t) &= \frac{\partial_s \hat{\xi}(s,t)}{\|\hat{\xi}(s)\|} - \frac{(\hat{\xi}(s), \partial_s \hat{\xi}(s))}{\|\hat{\xi}(s)\|^3} \hat{\xi}(s,t) \\
&= \frac{(P\varepsilon(s)\xi(s))(t)}{\|\hat{\xi}(s)\|} - \frac{(\eta(s), P\varepsilon(s)\xi(s))}{\|\hat{\xi}(s)\|} \eta(s,t).
\end{aligned} \tag{29}
$$

We note that

$$
\|\hat{\xi}(s) - \xi(s)\| \to 0
$$

as $s \to \infty$; otherwise we could find $\varepsilon > 0$ so that $\|\hat{\xi}(s) - \xi(s)\| \geqslant \varepsilon$ for all large s. But this would imply

$$
\varepsilon \leqslant \|(\mathrm{Id} - P)\xi(s_n)\| \to \|(\mathrm{Id} - P)e\| = 0.
$$

We also have $\|\hat{\xi}(s)\| \geqslant 1/2$ for all large s because

$$
\begin{aligned}
\|\hat{\xi}(s)\| &\geqslant \|\xi(s)\| - \|\xi(s) - \hat{\xi}(s)\| \\
&\geqslant \frac{3}{4} - \|\xi(s) - \hat{\xi}(s)\|.
\end{aligned}
$$

The last step follows from $\|\xi(s)\| \xrightarrow{s\to\infty} 1$:

$$\begin{aligned}\big|\|\xi(s)\|^2 - 1\big| &= \left|\int_0^T \langle \xi(s,t), -J_0 M_\infty(t)\xi(s,t)\rangle\, dt - 1\right| \\ &= \left|\int_0^T \langle \xi(s,t), J_0(M(v(s,t)) - M_\infty(t))\xi(s,t)\rangle\, dt\right| \\ &\leqslant \|\xi(s)\|_{L^2}^2 \sup_{0\leqslant t\leqslant T} |M(v(s,t)) - M_\infty(t)| \\ &\to 0 \quad \text{as } s \to \infty.\end{aligned}$$

We deduce from (29), using $(\eta(s), \partial_s\eta(s)) = 0$,

$$\begin{aligned}\|\partial_s\eta(s)\|^2 &= \left(\frac{P\varepsilon(s)\xi(s)}{\|\hat{\xi}(s)\|}, \partial_s\eta(s)\right) \\ &\leqslant \frac{2\,\|P\varepsilon(s)\xi(s)\|^2}{\|\hat{\xi}(s)\|^2} \\ &\leqslant c_0\, e^{-2\rho s}\end{aligned}$$

for a suitable constant c_0. Then

$$\begin{aligned}\|\eta(s_n) - \eta(\tau_n)\| &\leqslant \int_{\tau_n}^{s_n} \|\partial_s\eta(s)\|\, ds \\ &\leqslant \text{const}\cdot \int_{\tau_n}^{s_n} e^{-\rho s}\, ds \\ &\to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Note that $\|e\| = \|e'\| = 1$ since $\|\xi(s)\| \to 1$ as $s \to \infty$. Because of

$$\eta(s_n) = \frac{\hat{\xi}(s_n)}{\|\hat{\xi}(s_n)\|} \to \frac{e}{\|e\|},$$

$$\eta(\tau_n) = \frac{\hat{\xi}(\tau_n)}{\|\hat{\xi}(\tau_n)\|} \to \frac{e'}{\|e'\|},$$

we conclude that $e = e'$, which proves the lemma. □

Proof of Theorem 3.6. We know from Lemma 3.9 that

$$\begin{aligned}\zeta(s,t) &= \|\zeta(s)\|_s \xi(s,t) \\ &= e^{\int_{s_0}^{s} \alpha(\tau)\,d\tau} \|\zeta(s_0)\|_{s_0} \xi(s,t) \\ &= e^{\int_{s_0}^{s} \alpha(\tau)\,d\tau} [\tilde{e}(t) + r(s,t)],\end{aligned}$$

where

$$r(s,t) := \|\zeta(s_0)\|_{s_0} (\xi(s,t) - e(t)),$$

$$\tilde{e}(t) := \|\zeta(s_0)\|_{s_0} e(t) \in E,$$

and e is the eigenvector of A_∞ given by Lemma 3.12. We even have convergence of $\xi(s)$ to e in C^∞.

Let $n \in \mathbf{N}$ be an arbitrary number. By Lemma 3.10, the function $\xi(s)$ is bounded in $W^{n+2,2}([0,T],\mathbf{R}^2)$ independent of s. Then for every sequence $\tau_k' \to \infty$, there is a subsequence (τ_k) so that $\xi(\tau_k)$ converges in $W^{n+1,2}([0,T],\mathbf{R}^2)$, but since $\xi(s) \to e$ in $W^{1,2}([0,T],\mathbf{R}^2)$ already, we have $\xi(s) \to e$ in $W^{n+1,2}([0,T],\mathbf{R}^2)$ and finally in $C^n([0,T],\mathbf{R}^2)$ by the Sobolev embedding theorem. This shows that

$$|\partial_t^l r(s,t)| \to 0$$

as $s \to \infty$ uniformly in t for all $l \geqslant 0$. Recall that

$$\begin{aligned}\partial_s \xi(s,t) = {} & \big[M_\infty(t) - M(v(s,t))\big]\partial_t \xi(s,t) \\ & + (A_\infty - \lambda)\xi(s,t) + (\lambda - \alpha(s))\xi(s,t). \qquad (30)\end{aligned}$$

Also remember that

- $|\partial^\alpha (M_\infty(t) - M(v(s,t)))| \to 0$ as $s \to \infty$ uniformly in t for all $\alpha \in \mathbf{N}^2$, $|\alpha| \geqslant 0$,
- $\|(A_\infty - \lambda)\xi(s,t)\|_{C^l} \to 0$ as $s \to \infty$ since $\xi(s) \to e$ in C^{l+1}.

Applying ∂_t^l to (30) with an arbitrary integer $l \geqslant 0$, we obtain

$$|\partial_t^l \partial_s \xi(s,t)| \to 0$$

as $s \to \infty$ uniformly in t. We want to show that also $|\partial_t^l \partial_s^k \xi(s,t)| \to 0$ as $s \to \infty$ for all $k \geqslant 1$. Assume that this holds for some $k \geqslant 1$. If we apply $\partial_t^l \partial_s^k$ to (30), then

$$\begin{aligned}\partial_t^l \partial_s^{k+1} \xi(s,t) = {} & -\sum_{i=0}^{k} \sum_{j=0}^{l} \binom{l}{j} \binom{k}{i} \partial_s^{k-i} \partial_t^{l-j} [M(v(s,t))] \partial_s^i \partial_t^{j+1} \xi(s,t) \\ & - \sum_{i=0}^{k} \binom{k}{i} \frac{d^{k-i}\alpha}{ds^{k-i}}(s) \cdot \partial_s^i \partial_t^l \xi(s,t).\end{aligned}$$

We observe that all the expressions with $i \neq 0$ converge to zero by the induction hypothesis. If $i = 0$, we have to use that all the derivatives of α of order greater than 1 converge to zero. This is true because the derivatives of α are C^0-bounded, so we obtain C^∞_{loc}-convergence $\alpha(s) \to \lambda$ by the theorem of Ascoli and Arzela. □

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