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Localization and IDS Regularity in the Disordered Hubbard Model within Hartree–Fock Theory

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Abstract: Using the fractional moment method it is shown that, within the Hartree–Fock approximation for the disordered Hubbard Hamiltonian, weakly interacting Fermions at positive temperature exhibit localization, suitably defined as exponential decay of eigenfunction correlators. Our result holds in any dimension in the regime of large disorder and at any disorder in the one dimensional case. As a consequence of our methods, we are able to show Hölder continuity of the integrated density of states with respect to energy, disorder and interaction.

1. Introduction

Our goal in this note is to study Anderson localization in the context of infinitely many particles. We shall formulate our results for the disordered Hubbard model within Hartree–Fock theory. However, as the techniques involved are quite flexible, we expect that similar statements can be made in a more general framework, under appropriate modifications of the decorrelation estimates in Sect. 8.1. The (deterministic) Hubbard model under Generalized Hartree–Fock Theory has been discussed (at zero and positive temperature) by Lieb, Bach and Solovej in [22] but, to the best of our knowledge, the localization properties of the disordered version of this model remained unexplored, even in the context of restricted Hartree–Fock theory, up to the present work. The main difficulty lies in the addition of a self-consistent effective field, which will be random and non-local by nature, to a random Schrödinger operator. The conclusion of this note can be summarized as follows: under technical assumptions, the results on (single-particle) Anderson localization obtained in the non-interacting setting in the regimes of large disorder (in dimension $d \geq 2$) and at any disorder (in dimension $d = 1$), remain valid under the presence of sufficiently weak interactions. More specifically, in the regime of strong disorder this is accomplished in any dimension by Theorem 2 below. Theorem 1 contains the improvement in dimension one, where any disorder strength leads to localization, provided the interaction strength is taken sufficiently small. Our methods contain various

bounds on the fluctuations of the effective interaction which are interesting in their own right and potentially useful on different contexts. To exemplify this, we prove Hölder regularity of the integrated density of states (IDS) with respect to various parameters by adapting arguments of [18], which is the content of Theorem 3.

1.1. Discussion of the results and main obstacles. Mathematically, our setting can be understood as an Anderson-type model $H_\omega = H_0 + \lambda U_\omega$ where the values of the random potential U at different sites are correlated in a highly non-local and self-consistent fashion. The correlations are governed by a nonlinear function of H_ω, as explained on Sect. 3. In comparison to the recent result on Hartree–Fock theory for lattice fermions in [14], achieved via multiscale analysis, we use the fractional moment method to establish exponential decay of the eigenfunction correlators at large disorder in any dimension but also at any disorder in dimension one. In particular, in the above regimes we obtain for any $t > 0$, exponential decay (on expectation) for the matrix elements of the Hamiltonian evolution, which means that, on average, $|\langle m|e^{-itH}|n\rangle|$ decays exponentially on $|m-n|$.

The result of complete localization in dimension one in such interacting context is new and deserves attention on its own. Its main technical difficulty lies in the non-local correlations of the potential, which means that standard tools such as Furstenberg's theorem and Kotani theory are not available. Moreover, a large deviation theory for the Green's function is a further obstacle to establishing dynamical localization even if one obtains uniform positivity of the Lyapunov exponent. We overcome these challenges using ideas of [5, Chapter 12], where arguments reminiscent of the proof of the main result in [26] are presented. We then obtain positivity of the Lyapunov exponent at any disorder using uniform positivity for the Lyapunov exponent of the Anderson model, combined with an explicit bound on how this quantity depends on the interaction strength, see Theorem 8. When it comes to establishing a large deviation theorem, our modification of the argument in [5, Theorem 12.8] relies on quantifying the decorrelations of the effective potential, which is presented on Lemma 11 in the form of a strong mixing statement. It is worth clarifying that, since our proof is based on fractional moments, we have not established localization in one dimension for rough potentials as in [25]. Moreover, the gap assumption in [14] is replaced by working at positive temperature thus our results do not apply to Hartree–Fock ground states.

1.2. Hartree–Fock theory. Hartree–Fock theory has been widely applied in computational physics and chemistry. It also has a rich mathematical literature which goes well beyond the scope of random operators, see for instance [20–24] and references therein.

1.3. Background on localization for interacting systems. The main results of this note lie in between the vast literature on (non-interacting) single particle localization and the recent efforts to study many particle systems, as in the case of an arbitrary, but finite, number of particles in the series of works by Chulaevsky and Suhov [10–12] and Aizenman and Warzel [6]. In comparison to the later, we only seek for a single-particle localization result but allow for infinitely many interactions, which occur in the form of a mean field. In comparison to the recent developments on spin chains, as the study of the XY spin chain in [19] and the droplet spectrum of the XXZ quantum spin chain in [16] and [8], the notions of localization for a single-particle effective Hamiltonian are more clear and can be displayed from pure point spectrum to exponential decay of

eigenfunctions and exponential decay of eigenfunction correlators. The later is agreed to be the strongest form of single particle localization and it is what we accomplish in this manuscript. If fact, dynamical localization in the form of Theorems 1 and 2 implies pure point spectrum via the RAGE theorem (see [31, Proposition 5.3]) and exponential decay of eigenfunctions, see [5, Theorem 7.2 and Theorem 7.4].

2. Definitions and Statement of the Main Result

2.1. Notation. In what follows, $\mathbb{Z}^d$ will be equipped with the norm $|n| = |n_1| + \cdots + |n_d|$ for $n = (n_1, \ldots, n_d)$. Given a subset $\Lambda \subset \mathbb{Z}^d$, we define $\ell^2(\Lambda) := \{\varphi : \Lambda \to \mathbb{C} \mid \sum_{n \in \Lambda} |\varphi(n)|^2 < \infty\}$ and, for $\varphi \in \ell^2(\Lambda)$, we let $\|\varphi\|_{\ell^2(\Lambda)} := \left(\sum_{n \in \Lambda} |\varphi(n)|^2 < \infty\right)^{1/2}$. Throughout this note, η will be a positive constant and $F_{\beta,\kappa}$ will denote the Fermi-Dirac function at inverse temperature $\beta > 0$ and chemical potential κ:

$$F_{\beta,\kappa}(z) = \frac{1}{1 + e^{\beta(z-\kappa)}}. \tag{2.1}$$

We shall omit the dependence on the above parameters whenever it is clear from the context. For many of our bounds, the specific form of (2.1) is not important and F could denote a fixed function which is analytic on the strip $\mathcal{S} = \{z \in \mathbb{C} : |\mathrm{Im}\, z| < \eta\}$ and continuous up to the boundary of $\mathcal{S}$, in which case we define $\|F\|_\infty := \sup_{z \in \mathcal{S}} |F(z)|$. For the function $F_{\beta,\kappa}$ in (2.1) one can take $\eta = \frac{\pi}{2\beta}$. However, to obtain robust results which incorporate delicate fluctuations, further properties of the Fermi-Dirac function are necessary. Namely, in Sect. 8.1 we use the the the fact that $t F(t)$ is bounded as $t \to \infty$ and that $t(1 - F(t))$ is bounded $t \to -\infty$. These properties will also play a role in the decoupling estimates needed in the proof of Theorem 1 but could be relaxed if one is only interested in the large disorder proof of Theorem 2 for a specific distribution with heavy tails (for instance, the Cauchy distribution).

Our main goal is to study localization properties of non-local perturbations of the Anderson model $H_{\mathrm{And}} := -\Delta + \lambda V_\omega$ which naturally arise in the context of Hartree–Fock theory for the Hubbard model. The random potential V_ω is the multiplication operator on $\ell^2(\mathbb{Z}^d)$ defined as

$$(V_\omega \varphi)(n) = \omega_n \varphi(n) \tag{2.2}$$

for all $n \in \mathbb{Z}^d$ and $\{\omega_n\}_{n \in \mathbb{Z}^d}$ are independent, identically distributed random variables on which we impose technical assumptions described in the next paragraph. The hopping operator Δ is the discrete Laplacian on $\mathbb{Z}^d$, defined via

$$(\Delta \varphi)(n) = \sum_{|m-n|=1} (\varphi(m) - \varphi(n)). \tag{2.3}$$

The proofs of localization via fractional moments usually do not require the hopping to be dictated by Δ; below we will replace Δ by a more general operator H_0 whose matrix elements decay sufficiently fast away from the diagonal. It is technically useful to formulate some of our results in finite volume, i.e, we will work with restrictions of the operators to $\ell^2(\Lambda)$ but the estimates obtained will be volume independent, meaning that all the constants involved are independent of $\Lambda \subset \mathbb{Z}^d$. We will use $\mathbb{1}_\Lambda$ to denote the characteristic function of Λ as well as the natural projection $P_\Lambda : \ell^2(\mathbb{Z}^d) \to \ell^2(\Lambda)$. With these preliminaries we are ready to define the Schrödinger operators studied in this work.

2.2. Definition of the operators. Let $H_{\text{And}} = H_0 + \lambda V_\omega$ be the Anderson model on $\ell^2\left(\mathbb{Z}^d\right)$ where:

(A_1)

$$\zeta(\nu) := \sup_m \sum_{n \in \mathbb{Z}^d} |H_0(m, n)| \left(e^{\nu|m-n|} - 1\right) < \frac{\eta}{2}, \quad \text{for some } \nu > 0 \text{ fixed.}$$

(A_2) V_ω is defined as in (2.2) and the random variables $\{\omega(n)\}_{n \in \mathbb{Z}^d}$ are independent, identically distributed with a density ρ:

$$\mathbb{P}\left(\omega(0) \in I\right) = \int_I \rho(x)\, dx, \quad \text{for } I \subset \mathbb{R} \text{ a Borel set.}$$

(A_3) We also assume that $\operatorname{supp} \rho = \mathbb{R}$ with

$$\frac{\rho(x)}{\rho(x')} \geq e^{-c_1(\rho)|x-x'|(1+c_2(\rho)\max\{|x|,|x'|\})} \tag{2.4}$$

for some $c_1(\rho) > 0$ and $c_2(\rho) \geq 0$ and any $x, x' \in \mathbb{R}$.

Before stating the remaining assumptions on ρ, we need to introduce some notation. Assume that ρ satisfies (2.4). Let

$$\overline{\rho}(x) = \frac{\rho(x)}{\int_{-\infty}^{\infty} \rho(\alpha)h(x-\alpha)\, d\alpha} \tag{2.5}$$

where

$$h(x) = \begin{cases} e^{-\overline{c}_\rho|x|} & \text{if } c_2(\rho) = 0. \\[2mm] e^{-\overline{c}_\rho|x|^2} & \text{if } c_2(\rho) > 0. \end{cases} \tag{2.6}$$

(A_4) The function $\overline{\rho}$ is bounded for some $\overline{c}_\rho > 0$.

Remark. The technical assumptions $(A_3) - (A_4)$ will be needed for the large disorder result of Theorem 2 below. They include, for instance, the Cauchy distribution, the Gaussian, and the exponential distribution $\rho(v) = \frac{m}{2}e^{-m|v|}$.

The above assumptions will suffice to show localization at large disorder on Theorem 2 below. To show complete localization in dimension one, Theorem 1 will also require a moment condition on $\overline{\rho}$, which is the following.

(A_5) For some $\varepsilon > 0$ and some $\overline{c}_\rho > 0$, $\int_{-\infty}^{\infty} |x|^\varepsilon \overline{\rho}(x)\, dx < \infty$.

Remark. The assumption (A_5) covers, for example, the Gaussian and the exponential distributions but it does not cover the Cauchy or other distribution with heavy tails. It will be necessary for the one dimensional result of Theorem 1 below. More specifically, this requirement will imply a moment condition which will be needed to relate the Green's function to the Lyapunov exponent, see Sect. 5 Eq. (5.22).

Remark. The specific bound on $\zeta(\nu)$ is necessary to ensure that the Combes–Thomas bound $|G(m, n; E + i\eta)| \leq \frac{2}{\eta}e^{-\nu|m-n|}$ holds [5, Theorem 10.5], where $G(m, n; z)$ denotes $\langle m|(H - z)^{-1}|n\rangle$, whenever this quantity is defined.

Define the operator H_{Hub}, acting on $\ell^2\left(\mathbb{Z}^d\right) \oplus \ell^2\left(\mathbb{Z}^d\right)$ by

$$H_{\text{Hub}} = \begin{pmatrix} H_\uparrow(\omega) & 0 \\ 0 & H_\downarrow(\omega) \end{pmatrix} := \begin{pmatrix} H_0 + \lambda V_\omega + g V_\uparrow(\omega) & 0 \\ 0 & H_0 + \lambda V_\omega + g V_\downarrow(\omega) \end{pmatrix} \quad (2.7)$$

where the operators $H_\uparrow(\omega)$ and $H_\downarrow(\omega)$ act on $\ell^2\left(\mathbb{Z}^d\right)$ and the so-called effective potentials are defined via

$$\begin{pmatrix} V_\uparrow(\omega)(n) \\ V_\downarrow(\omega)(n) \end{pmatrix} = \begin{pmatrix} \langle n|F(H_\downarrow)|n\rangle \\ \langle n|F(H_\uparrow)|n\rangle \end{pmatrix}. \quad (2.8)$$

Note that the above equations only define $H_\uparrow(\omega)$ and $H_\downarrow(\omega)$ implicitly. Existence and uniqueness of $V_\uparrow$ and $V_\downarrow$ will be shown in Sect. 10.1 via a fixed point argument. The model (2.7) is usually referred to as the Hartree approximation, due to the absence of exchange terms. In Sect. 3 below we will show that the terminology Hartree–Fock approximation is justified when $g < 0$, which represents a repulsive interaction.

The Hubbard model is schematically represented in the following picture. The black (horizontal) edges represent hopping between sites and the red (vertical) edges represent the effective interaction between the two layers, which are identical copies of $\mathbb{Z}^d$.

$n_\uparrow$

$n_\downarrow$

$(0,0)$
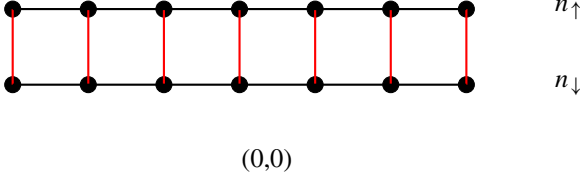
2.3. Main theorems. Fix an interval $I \subset \mathbb{R}$ and define the eigenfunction correlator through

$$Q_I(m, n) := \sup_{|\varphi|\leq 1} \left(|\langle m|\varphi(H_\uparrow)|n\rangle| + |\langle m|\varphi(H_\downarrow)|n\rangle|\right). \quad (2.9)$$

The operators $H_\uparrow$ and $H_\downarrow$ are defined as in (2.7) and the supremum is taken over Borel measurable functions bounded by one and supported on the interval I. In case $I = \mathbb{R}$ we simply write $Q(m, n)$. Our first result is the following:

Theorem 1. *In dimension $d = 1$, let $H_0 = -\Delta$ and assume that the conditions $(A_1) - (A_5)$ hold. For any $\lambda > 0$ and any closed interval $I \subset \mathbb{R}$, there is a constant $g_1 > 0$ such that whenever $|g| < g_1$ we have*

$$\mathbb{E}\left(Q_I(m, n)\right) \leq C e^{-\mu_1|m-n|}. \quad (2.10)$$

for any $m, n \in \mathbb{Z}^d$ and positive constants $\mu_1 = \mu_1(\lambda, \nu, \eta, I)$, $C(\eta, g, \lambda, \|F\|_\infty, I)$.

Theorem 2. *Suppose that the conditions $(A_1) - (A_4)$ hold. For any dimension $d \geq 1$, there exists a constant $g_d = g(d, \eta, \|F\|_\infty, \nu)$ such that, whenever $|g| < g_d$, there is a positive constant $\lambda_0(g)$ for which*

$$\mathbb{E}\left(Q(m, n)\right) \leq C e^{-\mu_d|m-n|}. \quad (2.11)$$

holds for $\lambda > \lambda_0(g)$, any $m, n \in \mathbb{Z}^d$ and some positive constants $\mu_d = \mu(d, \lambda, g, \nu, \eta)$, $C(\eta, \nu, d, g, \lambda, \|F\|_\infty)$.

Remark. It will follow from the proof that the constant g_d in Theorem 2 can be taken proportional to $\frac{\eta(1-e^{-\nu})^d}{\|F\|_\infty}$.

Remark. The constant g_1 in Theorem 1 can be taken equal to be the minimum among a factor proportional to $\frac{\eta(1-e^{-\nu})}{\|F\|_\infty}$ and the upper bound obtained in Corollary 9, which also depends on the lower bound for the Lyapunov exponent of the Anderson model on $\ell^2(\mathbb{Z})$.

Recall the definition of the integrated density of states for an ergodic operator H :

$$N_H(E) = \lim_{|\Lambda| \to \infty} \frac{\operatorname{Tr} P_{(-\infty, E)}(\mathbb{1}_\Lambda H \mathbb{1}_\Lambda)}{|\Lambda|}. \tag{2.12}$$

For the definition of ergodic operator one may consult [5, Definition 3.4]. In what follows, we denote by $N_0(E)$ the corresponding quantity for the free operator H_0 defined above, which is assumed to be ergodic for the result below, where we shall be concerned with the small disorder regime and aim for bounds which do not depend upon λ as $\lambda \to 0$.

Theorem 3. *Assume that $(A_1) - (A_2)$ hold with $x^2 \rho(x)$ bounded and that $g^2 < \lambda$. Fix a compact interval $K \subset \mathbb{R}$ where $E \mapsto N_0(E)$ is α_0-Hölder continuous and a bounded interval $J \subset \mathbb{R}$. The integrated density of states $N_{\lambda, g}(E)$ of H_{Hub} is Hölder continuous with respect to E and with respect to the pair (λ, g). More precisely:*

(IDS_1) For $E, E' \in K$

$$|N_{\lambda, g}(E) - N_{\lambda, g}(E')| \le C(\alpha, K)|E - E'|^\alpha \tag{2.13}$$

for $\alpha \in [0, \frac{\alpha_0}{2+\alpha_0}]$ and $C(\alpha, K)$ independent of λ and g.

(IDS_2) If $\lambda, \lambda' \in J$, we have that, for any $E \in K$, $\alpha \in [0, \frac{\alpha_0}{2+\alpha_0}]$ and $\beta \in [0, \frac{2}{\alpha+3d+4}]$,

$$|N_{\lambda, g}(E) - N_{\lambda', g'}(E)| \le C(\alpha_0, d, J, K)\left(|\lambda - \lambda'|^\beta + |g - g'|^\beta\right). \tag{2.14}$$

3. Motivation

We shall explain the motivation for the above choice of the effective potential. We are only going to outline the derivation of the self-consistent equations as this is a standard topic, see, for instance, [27, Chapter 3].

Let $\Lambda \subset \mathbb{Z}^d$ be a finite set. Following the notation of [22], we use Γ to denote a one particle density matrix, i.e, a 2×2 matrix whose entries are operators on $\ell^2(\Lambda)$ and which satisfies $0 \le \Gamma \le \mathbb{1}$. We then write

$$\Gamma := \begin{pmatrix} \Gamma_\uparrow & \Gamma_{\uparrow\downarrow} \\ \Gamma_{\downarrow\uparrow} & \Gamma_\downarrow \end{pmatrix}$$

where $\Gamma_{\downarrow\uparrow} = \Gamma_{\uparrow\downarrow}^\dagger$.

As in [22, Equation 3a.8], the pressure functional $\mathcal{P}(\Gamma)$ is defined as

$$-\mathcal{P}(\Gamma) = \mathcal{E}(\Gamma) - \beta^{-1}\mathcal{S}(\Gamma). \tag{3.1}$$

The energy functional is

$$\mathcal{E}(\Gamma) = \operatorname{Tr}\left(H_0 - \kappa + \lambda V_\omega\right)\Gamma + g \sum_n \langle n|\Gamma_\uparrow|n\rangle\langle n|\Gamma_\downarrow|n\rangle, \tag{3.2}$$

where we have identified $H_0 - \kappa + \lambda V_\omega$ with $\begin{pmatrix} H_0 - \kappa + \lambda V_\omega & 0 \\ 0 & H_0 - \kappa + \lambda V_\omega \end{pmatrix}$.

The entropy is given by

$$S(\Gamma) = -\mathrm{Tr}\left(\Gamma \log \Gamma + (1 - \Gamma) \log(1 - \Gamma)\right). \tag{3.3}$$

Generally, the choice of energy functional (3.2) is referred to as Hartree approximation as exchange terms are neglected. However, in the case of a repulsive interaction among the particles, it is easy to prove that such exchange terms do not affect the choice of minimizer for $-\mathcal{P}(\Gamma)$ and the process may be referred to as the Hartree–Fock approximation. Indeed, the Hartree–Fock energy for the repulsive interaction would incorporate the term $-g|\langle n|\Gamma_{\uparrow\downarrow}|n\rangle|^2$, which is non-negative when $g < 0$. Thus, for repusive interactions, off-diagonal terms can be disregarded for minimization purposes, see the analogous discussion in [22, Section 4a]. The minimizer Γ of $-\mathcal{P}(\Gamma)$ exists since Λ is a finite set. Moreover, it satisfies

$$\langle n|\Gamma_\uparrow|n\rangle = \langle n| \frac{1}{1 + e^{\beta(H_0 - \kappa + \lambda V_\omega + \mathrm{Diag}(\Gamma_\downarrow))}} |n\rangle. \tag{3.4}$$

$$\langle n|\Gamma_\downarrow|n\rangle = \langle n| \frac{1}{1 + e^{\beta(H_0 - \kappa + \lambda V_\omega + \mathrm{Diag}(\Gamma_\uparrow))}} |n\rangle. \tag{3.5}$$

Thus, the effective Hamiltonian on $\ell^2(\Lambda) \oplus \ell^2(\Lambda)$ is determined by

$$H_\omega^\Lambda := \begin{pmatrix} H_0 + \lambda\omega(n) + g V_\uparrow^\Lambda(n) & 0 \\ 0 & H_0 + \lambda\omega(n) + g V_\downarrow^\Lambda(n) \end{pmatrix}$$

$$V_\uparrow^\Lambda(\omega)(n) := \langle n| \frac{1}{1 + e^{\beta(H_0 - \kappa + \lambda\omega + g V_\downarrow)}} |n\rangle \tag{3.6}$$

$$V_\downarrow^\Lambda(\omega)(n) := \langle n| \frac{1}{1 + e^{\beta(H_0 - \kappa + \lambda\omega + g V_\uparrow)}} |n\rangle. \tag{3.7}$$

It will follow from arguments given below that if Λ_R is an increasing sequence with $\cup_{R\in\mathbb{N}}\Lambda_R = \mathbb{Z}^d$ then, for fixed $m \in \mathbb{Z}^d$,

$$\lim_{R\to\infty} V_{\mathrm{eff}}^{\Lambda_R}(m) = V_{\mathrm{eff}}(m) \tag{3.8}$$

and this fact ensures that, for localization purposes in the Hubbard model, it suffices to study H_{Hub} and its finite volume restrictions.

4. Outline of the Proof of Theorem 2

We now want to outline the proof of the Theorem 2 in the related model where H_{Hub} is replaced by the operator

$$H = H_0 + \lambda\omega(n) + g V_{\mathrm{eff}}(n) \tag{4.1}$$

acting on $\ell^2(\mathbb{Z}^d)$ with

$$V_{\mathrm{eff}}(n) = \langle n| \frac{1}{1 + e^{\beta(H_0 + \lambda\omega + g V_{\mathrm{eff}})}} |n\rangle. \tag{4.2}$$

In this case, the correlator is defined as

$$Q_I(m, n) := \sup_{|\varphi| \leq 1} |\langle m|\varphi(H)|n\rangle|. \tag{4.3}$$

where φ is Borel measurable and supported on I.

The above operator exhibits the main mathematical features of the Hubbard model, namely: the effective potential is defined self-consistently as a non-local and non-linear function of H. Thus, it is natural to first illustrate our methods here. For now let's assume the existence and uniqueness of V_{eff} are proven as well as its regularity with respect to $\{\omega(n)\}_{n\in\mathbb{Z}^d}$. Combined with estimates on the derivatives of V_{eff}, the above facts form a significant portion of the proof which is developed in Sects. 6 and 7. The, somewhat straightforward, extension of the proof to H_{Hub} will be explained in Sect. 10. A feature which Theorems 1 and 2 have in common is that the eigenfunction correlator decay will be achieved via the Green's function of $H^\Lambda = \mathbb{1}_\Lambda H \mathbb{1}_\Lambda$, which is H restricted to a finite set $\Lambda \subset \mathbb{Z}^d$. Let

$$G^\Lambda(m, n, z) = \langle m|(H^\Lambda - z)^{-1}|n\rangle. \tag{4.4}$$

Using the basics of the fractional moment method, which dates back to [1,4], we aim at showing that, for some $s \in (0, 1)$,

$$\mathbb{E}\left(\left|G^\Lambda(m, n; z)\right|^s\right) \leq Ce^{-\mu_d|m-n|} \tag{4.5}$$

holds uniformly in $z \in \mathbb{C}^+$, with positive constants $C = C(d, s, g, \lambda, \nu, \eta, \|F\|_\infty)$ and $\mu(d, s, g, \lambda, \nu, \eta, \|F\|_\infty)$ independent of the volume $|\Lambda|$. In this context, the Green's function decay expressed by Eq. (4.5) implies

$$\mathbb{E}(Q(m, n)) \leq C'e^{-\mu'_d|m-n|} \tag{4.6}$$

for some exponent $\mu'_d = \mu'(d, s, g, \lambda, \nu, \eta, \|F\|_\infty) > 0$ and $C' = C'(\eta, \nu, d, g, \lambda, s, \|F\|_\infty)$. This is well known and explained in great generality, for instance, in [2, Theorem A.1].

Another aspect which is shared by the proofs of Theorems 1 and 2 is that the starting point to obtain (4.5) will be the following *a-priori* bound where we let

$$U_\omega(n) = \omega(n) + \frac{g}{\lambda} V_{\text{eff}}(n, \omega). \tag{4.7}$$

be the "full" potential at site n and $\mathbb{E}_{\{U(m),U(n)\}}$ denotes the conditional expectation with respect to $\{U(m), U(n)\}$ at specified values of the remaining random variables.

Lemma 4. *Given a finite set $\Lambda \subset \mathbb{Z}^d$, there exist a constant $C_{\text{AP}} = C_{\text{AP}}(\eta, \nu, d, g, \lambda, s, \|F\|_\infty)$, independent of Λ, such that*

$$\mathbb{E}_{\{U(m),U(n)\}}\left(\left|G^\Lambda(m, n; z)\right|^s\right) \leq C_{\text{AP}} \tag{4.8}$$

holds for any $m, n \in \Lambda$, including the case $m = n$.

Lemma 4 follows from Lemma 5 below, see [5, Corollary 8.4]. From now on, to keep the notation simple, we drop the dependence on ω in the new variables $\{U(n)\}_{n\in\Lambda}$. Note that $U(n)$ and $U(m)$ are correlated for all values of m and n. The strategy is to show that, for g sufficiently small, they still behave as if they were independent in the following sense:

Lemma 5. *Fix $\Lambda \subset \mathbb{Z}^d$ finite and $n_0 \in \Lambda$. The conditional distribution of $U(n_0) = u$ at specified values of $\{U(n)\}_{n \in \Lambda \setminus \{n_0\}}$ has density $\rho_{n_0}^\Lambda$. Moreover, under assumptions $(A_1) - (A_4)$ we have that*

$$\sup_\omega \sup_\Lambda \sup_{n_0 \in \Lambda} \sup_{u \in \mathbb{R}} \rho_{n_0}^\Lambda(u) < \infty. \tag{4.9}$$

If, additionally, assumption (A_5) holds then $\rho_{n_0}^\Lambda(u) \in L^1(\mathbb{R}, |x|^\varepsilon dx)$, with moments uniformly bounded with respect to ω, Λ and n_0.

The proof of the above result is detailed in Sect. 9; it requires exponential decay of $|\frac{\partial V_{\text{eff}}(n)}{\partial \omega(m)}|$ and $|\frac{\partial^2 V_{\text{eff}}(n)}{\partial \omega(m) \omega(l)}|$ with respect to $|m - n|$ and $|m - n| + |l - n|$, respectively. The need for this decay is the main reason to require $\beta > 0$ or, in other words, to require analiticity of F on a strip. The intuitive explanation for Lemma 5 is that the random variables $U(n)$ and $U(n_0)$ decorrelate in a strong fashion as $|n - n_0|$ becomes large. Lemma 5 implies (4.5) for any $0 < s < 1$ as long as λ is taken sufficiently large, see [5, Theorem 10.2].

The proof of Theorem 1 will require additional efforts involving tools which are specific to one dimension, which we shall comment on below.

5. One Dimensional Aspects: Strategy of the Proof of Theorem 1

The argument for proving Theorem 1 follows closely the approach in the proof of theorem 12.11 in [5], which we now recall.

5.1. Main ideas in the i.i.d case. In the reference [5, Chapter 12] Green's function decay is described in terms of the moment generating function, defined by

$$\varphi(s, z) = \lim_{|n| \to \infty} \frac{\ln \mathbb{E}(|G(0, n; z)|^s)}{|n|}. \tag{5.1}$$

The existence of the above quantity for all $z \in \mathbb{C}^+$ and $s \in (0, 1)$ and its relationship to the Lyapunov exponent are a consequence of Fekete's lemma [5, Proposition 12.9]:

Lemma 6 (Fekete). *Let $\{a_n\}_{n \in \mathbb{N}}$ be a sequence of real numbers such that, for every pair (m, n) of natural numbers,*

$$a_{n+m} \leq a_n + a_m \tag{5.2}$$

Then, $\alpha = \lim_{n \to \infty} \frac{a_n}{n}$ exists and equals $\inf_{n \in \mathbb{N}} \frac{a_n}{n}$.

It is an elementary observation that if, instead, the sequence $\{a_n\}_{n \in \mathbb{N}}$ satisfies $a_{n+m} \leq a_n + a_m + C$ then the above result applies to $b_n := a_n + C$ and that an analogous statement holds for superadditive sequences, which satisfy (5.2) with the inequality reversed. In the i.i.d. context, the sequence $a_n = \ln \mathbb{E}(|G(0, n; z)|^s)$ is shown to be both subbaditive and superadditive, meaning that there exist constants $C_-(s, z)$ and $C_+(s, z)$ for which

$$a_n + a_m + C_- \leq a_{n+m} \leq a_n + a_m + C_+. \tag{5.3}$$

holds for all $m, n \in \mathbb{N}$, see [5, Lemma 12.10]. A consequence of this fact, together with a precise control of the arising constants, is stated below.

Lemma 7 [5, Theorem 12.8]. *For any $z \in \mathbb{C}^+$, there are $c_s(z), C_s(z) \in (0, \infty)$ such that for all $n \in \mathbb{Z}$*

$$c_s^{-1}(z)e^{\varphi(s,z)|n|} \leq \mathbb{E}\left(|G_{\mathrm{And}}(0, n; z)|^s\right) \leq C_s(z)e^{\varphi(s,z)|n|}. \tag{5.4}$$

Moreover, for any compact set $K \subset \mathbb{R}$ and $S \subset [-1, 1)$, we have the local uniform bound

$$\sup_{s \in S} \sup_{z \in K + i(0,1]} \max\{c_s(z), C_s(z)\} < \infty \tag{5.5}$$

and the same result holds with z replaced by its boundary value $E + i0$ for Lebesgue almost every E.

On the other hand, for fixed $z \in \mathbb{C}^+$, $\varphi(s, z)$ is shown to be convex function of s and non-increasing in $[-1, +\infty)$, with its derivative at $s = 0$ satisfying $\frac{\partial \varphi(0,z)}{\partial s} = -\mathcal{L}(z)$. It is a consequence of these facts that for almost every $E \in \mathbb{R}$ there exists a value $s = s(E) \in (0, 1)$ such that

$$\varphi(s, E) \leq -\frac{s}{2}\mathcal{L}(E). \tag{5.6}$$

The above is the content of [5, Equation (12.86)]. Dynamical localization is shown to hold locally as a consequence of the inequality (5.3) along with Lemma 7, the inequality (5.6) and Kotani theory, which establishes that $\mathcal{L}(E)$ is positive for almost every $E \in \mathbb{R}$.

5.2. Modifications. In this section we will outline the proof of Theorem 1 with H_{Hub} again replaced by the operator H on $\ell^2\left(\mathbb{Z}^d\right)$ defined in (4.1). For simplicity we set $\lambda = 1$ since the disorder strength does not play an important role in Theorem 1. Let $H^+ = H_{[0,\infty)\cap\mathbb{Z}}$ be the restriction of H to $\ell^2\left(\mathbb{Z}^+\right)$ and denote by $G^+(m, n; z)$ the Green's function of H^+. Recall the definition of the Lyapunov exponent: initially, for $z \in \mathbb{C}^+$, we let

$$\mathcal{L}(z) = -\mathbb{E}\left(\ln|G^+(0, 0; z)|\right). \tag{5.7}$$

By Herglotz theory (see, for instance, [5, Appendix B] and references therein) it is seen that, for Lebesgue almost every $E \in \mathbb{R}$, $\mathcal{L}(E)$ is well defined as $\lim_{\delta \to 0^+} \mathcal{L}(E + i\delta)$. Finally, recall the uniform positivity of the Lyapunov exponent for the Anderson model on $\ell^2\left(\mathbb{Z}\right)$:

$$\operatorname*{ess\,inf}_{E \in \mathbb{R}} \mathcal{L}_{\mathrm{And}}(E) > \mathcal{L}_{\mathrm{And}} \tag{5.8}$$

for some $\mathcal{L}_{\mathrm{And}} > 0$. The first step towards Green's function decay (4.5) will be showing uniform positivity of $\mathcal{L}(E)$, which is accomplished by the following.

Theorem 8. *There exists a constant $C_{\mathrm{Lyap}}(s, \eta, g, \|F\|_\infty) > 0$ such that*

$$|\mathcal{L}(z) - \mathcal{L}_{\mathrm{And}}(z)| \leq C_{\mathrm{Lyap}}|g|^s \tag{5.9}$$

for all $z \in \mathbb{C}^+$.

Proof. From the resolvent identity we obtain

$$\frac{|G^+(0,0;z)|}{|G^+_{\text{And}}(0,0;z)|} \leq 1 + |g|\|F\|_\infty \sum_n |G^+(0,n;z)| \frac{|G^+_{\text{And}}(n,0;z)|}{|G^+_{\text{And}}(0,0;z)|} \tag{5.10}$$

$$\frac{|G^+_{\text{And}}(0,0;z)|}{|G^+(0,0;z)|} \leq 1 + |g|\|F\|_\infty \sum_n |G^+_{\text{And}}(0,n;z)| \frac{|G^+(n,0;z)|}{|G^+(0,0;z)|} \tag{5.11}$$

Using the bound $\ln(1+x) \leq \frac{x^s}{s}$ for $0 < s < 1$ and $x > 0$ we reach, for $0 < s < 1/2$,

$$\ln\left(\frac{|G^+(0,0;z)|}{|G^+_{\text{And}}(0,0;z)|}\right) \leq \frac{|g|^s}{s}\|F\|_\infty^s \sum_n |G^+(0,n;z)|^s \frac{|G^+_{\text{And}}(n,0;z)|^s}{|G^+_{\text{And}}(0,0;z)|^s}. \tag{5.12}$$

Taking expectations, using the definition of the Lyapunov exponents and the Cauchy–Schwarz inequality

$$\mathcal{L}_{\text{And}}(z) - \mathcal{L}(z) \leq \frac{|g|^s}{s}\|F\|_\infty^s \sup_n \mathbb{E}\left(|G^+(0,n;z)|^{2s}\right)^{1/2} \sum_n \mathbb{E}\left(\frac{|G^+_{\text{And}}(n,0;z)|^{2s}}{|G^+_{\text{And}}(0,0;z)|^{2s}}\right)^{1/2}$$

$$:= C_{\text{Lyap}}(s,\eta,\nu,\|F\|_\infty)|g|^s. \tag{5.13}$$

The fact that C_{Lyap} is a finite quantity follows from a couple of remarks. Firstly, by Feenberg's expansion [5, Theorem 6.2] we have the identity

$$|G^+_{\text{And}}(n,0;z)| = |G^+_{\text{And}}(0,0;z)||G^+_{\text{And}}(1,n;z)| \tag{5.14}$$

where $G^+_{\text{And}}(1,n;z)$ denotes the Green's function of H_{And} restricted to $\ell^2(\mathbb{Z}) \cap [1,\infty)$. From the *a-priori* fractional moment bound on Lemma 4 combined with the Green's function decay for the one dimensional Anderson model

$$\mathbb{E}\left(|G^+_{\text{And}}(1,n;z)|^{2s}\right) < C(s)e^{-\mu_{\text{And}}|n|} \tag{5.15}$$

we conclude that that $C_{\text{Lyap}} < \infty$. The estimate for $\mathcal{L}(z) - \mathcal{L}_{\text{And}}(z)$ is similar. □

In principle one might worry that the pre-factor C_{Lyap} on the above bound will depend on g. However, it is easy to see from the arguments in the proof of Lemma 4, that C_{AP} converges to a finite quantity as $g \to 0$, thus we shall disregard its dependence on g.

Corollary 9. *Whenever* $|g| < \left(\frac{\mathcal{L}_{\text{And}}}{C_{\text{Lyap}}}\right)^{1/s}$ *holds for some* $s \in (0,1/2)$, *we have*

$$\mathcal{L}_0 := \operatorname*{ess\,inf}_{E \in \mathbb{R}} \mathcal{L}(E) > 0. \tag{5.16}$$

We can now proceed to the second step of the proof of Theorem 1, which consists of establishing Green's function decay from Corollary 9. For that purpose, an important detail to keep in mind is that, in the correlated context, if we choose $a_n = \log \mathbb{E}(|G(0,n;z)|^s)$, the condition (5.3) will not be fulfilled for all pairs (m,n) due to the lack of independence between the potentials. This means that Fekete's lemma is not applicable. Moreover, its well-studied modifications (for instance by Erdös and de Bruijn [15]) do not seem to suffice either.

To the best of our knowledge the result given below is new. Its formulation takes into account the strong decorrelation between the potentials in the Hubbard model and introduces a notion of approximate subbaditivity.

Lemma 10 (Fekete-type lemma for approximately subbaditive sequences). *Let $\delta > 0$ be given and $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers such that, for every triplet m, n, r of natural numbers with $r \geq \delta \max\{\log m, \log n\}$, the inequality*

$$a_{n+m+r} \leq a_n + a_m + C \tag{5.17}$$

holds with a constant C independent of m, n and r. Then,

$$\alpha = \lim_{n\to\infty} \frac{a_n}{n} \tag{5.18}$$

exists and equals $\inf_{n\in\mathbb{N}} \frac{a_n + C}{n}$. Moreover, $\alpha \in [-\infty, 0]$.

Note that, as a consequence, we have

$$a_n \geq n\alpha - C \tag{5.19}$$

for all $n \in \mathbb{N}$, where C is the same constant as in (5.17). The following decoupling estimate guarantees the applicability of the above lemma with the choice $a_n = \log \mathbb{E}\left(|\hat{G}(0, n; z)|^s\right)$, where $\hat{G}(0, n; z) = \langle 0|(H_{[0,n]} - z)^{-1}|n\rangle$ is the Green's function of the operator H restricted to $\ell^2([0, n] \cap \mathbb{Z})$

Lemma 11 (Strong mixing decoupling). *There exist positive numbers $C_{\mathrm{Dec}}(s, v, \eta, g, \|F\|_\infty)$, and $\delta = \delta(\eta, v, g, \|F\|_\infty)$ such that the inequality*

$$\mathbb{E}\left(|\hat{G}(0, n+m+r; z)|^s\right) \leq C_{\mathrm{Dec}}^{r+1}\mathbb{E}\left(|\hat{G}(0, n; z)|^s\right)\mathbb{E}\left(|\hat{G}(0, m; z)|^s\right) \tag{5.20}$$

holds whenever $r \geq \delta \log \max\{m, n\}$.

The above implies that the sequence $b_n = -n \ln C_{\mathrm{Dec}} + \ln \mathbb{E}\left(|\hat{G}(0, n; z)|^s\right)$ satisfies $b_{n+m+r} \leq b_n + b_m + \ln C_{\mathrm{Dec}}$. In particular $\lim_{n\to\infty} \frac{b_n}{n}$ exist and so does

$$\varphi(s, z) = \lim_{|n|\to\infty} \frac{\ln \mathbb{E}\left(|\hat{G}(0, n; z)|^s\right)}{|n|}. \tag{5.21}$$

After obtaining an analogue of Lemma 7, the final step will be to relate the moment-generating function to the Lyapunov exponent through an inequality of the type

$$\varphi(s, E) \leq -\frac{s}{2}\mathcal{L}_0. \tag{5.22}$$

In particular, $\varphi(s, E) \leq 0$. In reference [5], the bound (5.22) is stated with $\mathcal{L}_0$ replaced by $\mathcal{L}(E)$ and with s depending on E. However, it is easy to see from the arguments given there that s can be chosen locally uniformly in E, in which case Corollary 9 allows us to conclude (5.22). For further details, see [5, Equations (12.79) and (12.80)].
The mere existence of the limit in (5.21), along with the fact that $\varphi < 0$, implies that for $|n|$ sufficiently large we have

$$\mathbb{E}\left(|\hat{G}(0, n; z)|^s\right) \leq e^{\frac{\varphi(s,z)}{2}|n|}. \tag{5.23}$$

The above, combined with (5.22), gives, for $|n|$ sufficiently large,

$$\mathbb{E}\left(|\hat{G}(0, n; z)|^s\right) \le e^{\frac{-s\mathcal{L}_0}{4}|n|}. \tag{5.24}$$

Hence, there exists a positive constant C which is locally uniform in z and such that

$$\mathbb{E}\left(|\hat{G}(0, n; z)|^s\right) \le Ce^{\frac{-s\mathcal{L}_0}{4}|n|} \tag{5.25}$$

holds for all $n \in \mathbb{Z}$.

Making use of Lemma 4 and translation invariance we reach, up to an adjustment of the constant C,

$$\mathbb{E}\left(|G(0, n; z)|^s\right) \le Ce^{\frac{-s\mathcal{L}_0}{4}|n|}. \tag{5.26}$$

for all $n \in \mathbb{Z}$. This finishes the proof of Theorem 1 assuming Lemmas 10 and 11 , which we prove later in Sect. 11.

The remainder of the paper is organized as follows. Sections 6, 7 and 8 establish existence, regularity and decay properties of the effective potential in the model (4.1). Section 9 contains the proof of Lemma 5. Section 10 explains the required modifications for the Hubbard model. The proof of Theorem 2 is finished combining Sects. 9 and 10. Section 11 covers the remaining details of the proof of Theorem 1 and Sect. 12 contains the proof of Theorem 3.

6. Existence of the Effective Potential

To justify the definition of the effective potential in (4.1), let $\Phi(V) : \ell^\infty(\mathbb{Z}^d) \to \ell^\infty(\mathbb{Z}^d)$ be given by $\Phi(V)(n) := \langle n | F(T + \lambda V_\omega + gV) | n \rangle$. Recall that F is analytic, bounded on the strip $S = \{|\mathrm{Im} z| < \eta\}$ and continuous up to the boundary of S. Our goal is to check that Φ is a contraction on $\ell^\infty(\mathbb{Z}^d)$, meaning that

$$\|\Phi(V) - \Phi(W)\|_{\ell^\infty(\mathbb{Z}^d)} < c\|V - W\|_{\ell^\infty(\mathbb{Z}^d)} \tag{6.1}$$

holds for some $c < 1$ and all $V, W \in \ell^\infty(\mathbb{Z}^d)$. Using the analiticity of F we have the following representation [3, Equation (D.2)]

$$F(T + \lambda V_\omega + gV) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left(\frac{1}{T + \lambda V_\omega + gV - i\eta + t} - \frac{1}{T + \lambda V_\omega + gV + i\eta + t} \right) f(t)\, dt \tag{6.2}$$

for all $V \in \ell^\infty(\mathbb{Z}^d)$, where $f = F_+ + F_- + D * F$ for $F_\pm(u) = F(u \pm i\eta)$ and $D(u) = \frac{\eta}{\pi(\eta^2 + u^2)}$ is the Poisson kernel. It follows immediately that $\|f\|_\infty \le 3\|F\|_\infty$. This is a prelude for the following fixed point argument, where the operator T will be assumed to satisfy

$$\sup_n \sum_m |T(m, n)| \left(e^{\nu|m-n|} - 1 \right) < \frac{\eta}{2}. \tag{6.3}$$

Proposition 12. (C_1) *For any self-adjoint operator T on $\ell^2\left(\mathbb{Z}^d\right)$ satisfying (6.3) and bounded potentials V, W, we have, for any $v' \in (0, v)$, that*

$$\left|\langle m|(F(T+V) - F(T+W))|n\rangle\right| \leq \frac{72\sqrt{2}e^{-v'|m-n|}}{\eta\left(1 - e^{v'-v}\right)^d}\|F\|_\infty\|V - W\|_\infty. \quad (6.4)$$

(C_2) *For any self-adjoint operator T on $\ell^2\left(\mathbb{Z}^d\right) \oplus \ell^2\left(\mathbb{Z}^d\right)$ satisfying (6.3) and bounded potentials V, W on $\ell^2\left(\mathbb{Z}^d\right) \oplus \ell^2\left(\mathbb{Z}^d\right)$ we have, for any $v' \in (0, v)$, that*

$$\left|\langle m|(F(T+V) - F(T+W))|n\rangle\right| \leq \frac{144\sqrt{2}e^{-v'|m-n|}}{\eta\left(1 - e^{v'-v}\right)^d}\|F\|_\infty\|V - W\|_\infty \quad (6.5)$$

(C_3) *For any $m, n, j \in \mathbb{Z}^d$, the matrix elements $\langle m|F(T + gV)|n\rangle$ are differentiable with respect to $V(j)$ and*

$$\left|\frac{\partial\langle m|F(T+gV)|n\rangle}{\partial V(j)}\right| \leq |g|\frac{72\sqrt{2}e^{-v(|m-j|+|n-j|)}}{\eta}\|F\|_\infty\|V\|_\infty. \quad (6.6)$$

Proof. The resolvent identity gives

$$\langle m|\frac{1}{T+V-t-i\eta} - \frac{1}{T+W-t-i\eta}|n\rangle + \langle m|\frac{1}{T+W-t+i\eta} - \frac{1}{T+V-t+i\eta}|n\rangle$$

$$= \langle m|(\frac{1}{T+V-t-i\eta} - \frac{1}{T+V-t+i\eta})(W-V)\frac{1}{T+W-t-i\eta}|n\rangle$$

$$- \langle m|\left(\frac{1}{T+W-t+i\eta} - \frac{1}{T+W-t-i\eta}\right)(W-V)\frac{1}{T+V-t+i\eta}|n\rangle.$$

Taking absolute values in the first term on the right-hand side we obtain

$$\left|\langle m|\left(\frac{1}{T+V-t-i\eta} - \frac{1}{T+V-t+i\eta}\right)(W-V)\frac{1}{T+W-t-i\eta}|n\rangle\right|$$

$$\leq \sum_{l\in\mathbb{Z}^d}|G^V(m,l;t+i\eta) - G^V(m,l;t-i\eta)||(W-V)(l)|G^W(l,n;t+i\eta)|$$

$$\leq 24\sum_l|(V-W)(l)|e^{-v(|l-n|+|l-m|)}\langle m|\frac{1}{(T+V-t)^2+\eta^2/2}|m\rangle^{1/2}\langle l|\frac{1}{(T+V-t)^2+\eta^2/2}|l\rangle^{1/2}.$$

In the last step we made use of the Combes–Thomas bound $|G^W(m,n;t+i\eta)| \leq \frac{2}{\eta}e^{-v|m-n|}$ as well as lemma 3 in [3, appendix D] to estimate the difference between the Green functions as

$$|G^V(m,l;t+i\eta) - G^V(m,l;t-i\eta)|$$

$$\leq 12\eta e^{-v|m-l|}\langle m|\frac{1}{(T+V-t)^2+\eta^2/2}|m\rangle^{1/2}\langle l|\frac{1}{(T+V-t)^2+\eta^2/2}|l\rangle^{1/2}. \quad (6.7)$$

Integrating over t we conclude, using Cauchy–Schwarz and the spectral measure representation, that

$$\int_{-\infty}^{\infty} \left| \langle m | \left(\frac{1}{T + V - t - i\eta} - \frac{1}{T + V - t + i\eta} \right) (W - V) \frac{1}{T + W - t - i\eta} | n \rangle \right| dt$$

(6.8)

$$\leq \frac{24\sqrt{2}\pi}{\eta} \sum_l |(V - W)(l)| e^{-\nu(|l-n|+|l-m|)}.$$

The above implies that

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \langle m | \left(\frac{1}{T + V - t - i\eta} - \frac{1}{T + V - t + i\eta} \right) (W - V) \frac{1}{T + W - t - i\eta} | n \rangle \right| dt$$

$$\leq \frac{12\sqrt{2}}{\eta} \|V - W\|_\infty e^{-\nu'|m-n|} \sum_{l \in \mathbb{Z}^d} e^{(\nu'-\nu)|l-n|}$$

$$= \frac{12\sqrt{2}}{\eta} \|V - W\|_\infty e^{-\nu'|m-n|} \frac{1}{\left(1 - e^{\nu'-\nu}\right)^d}.$$

As a similar bound holds for $\frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \langle m | \left(\frac{1}{H_0 + W - t + i\eta} - \frac{1}{H_0 + W - t - i\eta} \right) (V - W) \frac{1}{H_0 + V - t + i\eta} | n \rangle \right| dt$, we conclude the proof of the inequality (6.4) by recalling that $\|f\|_\infty \leq 3\|F\|_\infty$. The inequality (6.5) in the statement of Proposition 12 follows from the same argument with the only difference that one has to sum two geometric series, hence the modification on the upper bound. The bound (6.6) is proven similarly: note that $h \frac{12\sqrt{2}\pi}{\eta} e^{-\nu|j-n|} e^{-\nu|m-j|}$ is an upper bound for the left-hand side of Eq. (6.8) with V replaced by gV and $W = g(V + hP_j)$, where P_j denotes the projection onto $\mathrm{Span}\{\delta_j\}$. We also observe that this time there will be no summation over l, hence the introduction of ν' is unnecessary. We then conclude that

$$\left| \frac{\langle m | F(T + gV + hP_j) | n \rangle - \langle m | F(T + gV) | n \rangle}{h} \right| \leq \frac{72\sqrt{2}\pi}{\eta} e^{-\nu|j-n|} e^{-\nu|m-j|}. \quad (6.9)$$

Letting $h \to 0$ finishes the proof. □

Taking $m = n$, as a consequence of the above Proposition, (6.1) holds whenever

$$|g| < \frac{\eta \left(1 - e^{-\nu}\right)^d}{72\sqrt{2}\|F\|_\infty}. \quad (6.10)$$

This observation yields the following.

Proposition 13. *Let* $g_d = \frac{\eta(1-e^{-\nu})^d}{72\sqrt{2}\|F\|_\infty}$. *Then, for* $|g| < g_d$, *there is a unique effective potential* $V_{\mathrm{eff}} \in \ell^\infty \left(\mathbb{Z}^d\right)$ *satisfying*

$$V_{\mathrm{eff}}(n) = \langle n | F(H_0 + \lambda\omega + gV_{\mathrm{eff}} | n \rangle. \quad (6.11)$$

Moreover, for $\Lambda \subset \mathbb{Z}^d$, *there is a unique* V_{eff}^Λ *in* $\ell^2 (\Lambda)$ *satisfying* (6.11) *with H replaced by* $H^\Lambda = \mathbb{1}_\Lambda H \mathbb{1}_\Lambda$.

Remark. Replacing H_0 by $H_0 - \kappa I$ we can incorporate a chemical potential in our results. For simplicity, we shall make no further reference to κ during the proofs and assume it was already incorporated to H_0.

7. Regularity of the Effective Potential

Our goal in this section is to conclude that, for a fixed finite subset $\Lambda \subset \mathbb{Z}^d$ with $|\Lambda| = n$, the effective potential V_{eff} is a smooth function of $\{\omega(j)\}_{j \in \Lambda}$. This will be of relevance for several resampling arguments later in the note. For that purpose, define a map $\xi : \ell^\infty\left(\mathbb{Z}^d\right) \times \mathbb{R}^n \to \ell^\infty\left(\mathbb{Z}^d\right)$ by

$$\xi(V, \omega)(j) = V(j) - \langle j | F(H_0 + \lambda \omega + gV) | j \rangle \qquad (7.1)$$

Then, V_{eff} is the unique solution of $\xi(V, \omega) = 0$. Thus, its regularity can be inferred via the implicit function theorem once we check that the derivative $D\xi(\cdot, \omega)$ is non-singular. Note that

$$\frac{\partial \xi(V, \omega)(j)}{\partial V(l)} = \delta_{jl} - \frac{\partial \langle j | F(H_0 + \lambda \omega + gV) | j \rangle}{\partial V(l)}. \qquad (7.2)$$

Using Lemma 12, we have that

$$\left| \frac{\partial \langle j | F(H_0 + \lambda \omega + gV) | j \rangle}{\partial V(l)} \right| \leq |g| \frac{72\sqrt{2} e^{-2\nu|j-l|}}{\eta} \|F\|_\infty. \qquad (7.3)$$

In particular, whenever $|g| \frac{72\sqrt{2}\|F\|_\infty}{\eta(1-e^{-2\nu})^d} < 1$ we have that the operator $D\xi(\omega, .) : \ell^\infty(\Lambda) \to \ell^\infty(\Lambda)$ is invertible since it has the form $I + gM$ where, by (6.6), gM has operator norm less than one. Note the smallness condition on g is independent of $\Lambda \subset \mathbb{Z}^d$. It is a consequence of the implicit function theorem that V is a smooth function of $(\omega(1), \ldots, \omega(n))$.

8. Decay Estimates for the Effective Potential

We start this section with the following lemma, which will be useful to formulate the decay of correlations between $U(n)$ and $U(m)$ as $|m - n| \to \infty$.

Lemma 14. *Whenever $\frac{72\sqrt{2}|g|\|F\|_\infty}{\eta(1-e^{-\nu})^d} < 1$, there exist constants $C_1(d, \lambda, g, \eta, \|F\|_\infty, \nu)$ and $C_2(d, \lambda, g, \eta, \|F\|_\infty, \nu)$ such that*

$$\max\left\{ \sum_m e^{\nu|n-m|} \left| \frac{\partial V_{\text{eff}}(n)}{\partial \omega(m)} \right|, \sum_n e^{\nu|n-m|} \left| \frac{\partial V_{\text{eff}}(n)}{\partial \omega(m)} \right| \right\} \leq C_1 \qquad (8.1)$$

$$\sum_{l,m,n} e^{\nu(|l-n|+|n-m|+|l-m|)} \left| \frac{\partial^2 V_{\text{eff}}(n)}{\partial \omega(m) \partial \omega(l)} \right| \leq C_2. \qquad (8.2)$$

Moreover C_1 and C_2 can be bounded from above by a constant of the form $\frac{\lambda D}{1-g\theta}$ with D and θ independent of g and these constants are explicit in the proof.

Proof. For convenience we denote $V_{\text{eff}} = V$. As in Sect. 6 we write $F(H) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left(\frac{1}{H+t-i\eta} - \frac{1}{H+t+i\eta} \right) f(t)\, dt$ where f is bounded by $3\|F\|_{\infty}$. Thus

$$V(n, \omega) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} K(n, t, \omega) f(t)\, dt \tag{8.3}$$

where $K(n, t, \omega) = G(n, n; t - i\eta) - G(n, n; t + i\eta)$. Denote by P_m the projection mapping $\ell^2\left(\mathbb{Z}^d\right)$ onto $\ell^2\left(\text{Span}\{\delta_m\}\right)$. Using difference quotients, it is easy to check

$$\frac{\partial}{\partial \omega(m)} \frac{1}{H-z} + g \frac{1}{H-z} \frac{\partial V}{\partial \omega(m)} \frac{1}{H-z} = -\lambda \frac{1}{H-z} P_m \frac{1}{H-z}. \tag{8.4}$$

Taking matrix elements we obtain

$$\frac{\partial K(n, t, \omega)}{\partial \omega(m)} = -g \sum_l \tilde{G}(l, n) \frac{\partial V(l)}{\partial \omega(m)} + \lambda r(m, n).$$

$$\tilde{G}(l, n) := G(l, n; t + i\eta) G(n, l; t + i\eta) - G(l, n; t - i\eta) G(n, l; t - i\eta).$$

$$r(m, n) := G(n, m; t + i\eta) G(m, n; t + i\eta) - G(n, m; t - i\eta) G(m, n; t - i\eta).$$

Note that

$$\tilde{G}(l, n) = \left(G(l, n; t + i\eta) - G(l, n; t - i\eta) \right) G(n, l; t + i\eta)$$
$$+ \left(G(n, l; t + i\eta) - G(n, l; t - i\eta) \right) G(l, n; t - i\eta). \tag{8.5}$$

We now make use of [3, Lemma 3]:

$$|G(l, n; t + i\eta) - G(l, n; t - i\eta)|$$
$$\leq 12\eta e^{-v|l-n|} \langle n| \frac{1}{(H-t)^2 + \eta^2/2} |n\rangle^{1/2} \langle l| \frac{1}{(H-t)^2 + \eta^2/2} |l\rangle^{1/2}. \tag{8.6}$$

This, together with the Combes-Thomas bound $|G(l, n, t \pm i\eta)| \leq \frac{2}{\eta} e^{-v|l-n|}$ and (8.5) implies

$$|\tilde{G}(l, n)| \leq 48 e^{-2v|l-n|} \langle n| \frac{1}{(H-t)^2 + \eta^2/2} |n\rangle^{1/2} \langle l| \frac{1}{(H-t)^2 + \eta^2/2} |l\rangle^{1/2}.$$

$$|r(m, n)| \leq 48 e^{-2v|m-n|} \langle m| \frac{1}{(H-t)^2 + \eta^2/2} |m\rangle^{1/2} \langle n| \frac{1}{(H-t)^2 + \eta^2/2} |n\rangle^{1/2}.$$

Thus

$$\tilde{K}(l, n) := \int_{-\infty}^{\infty} |\tilde{G}(l, n)|\, dt \leq \frac{48\sqrt{2}\pi}{\eta} e^{-2v|l-n|}$$

$$\tilde{r}(m, n) := \int_{-\infty}^{\infty} |r(m, n)|\, dt \leq \frac{48\sqrt{2}\pi}{\eta} e^{-2v|m-n|}.$$

To summarize, we have shown the following inequality

$$\left| \frac{\partial V(n)}{\partial \omega(m)} \right| \leq \frac{3\|F\|_{\infty}}{2\pi} \left(|g| \sum_l \tilde{K}(l, n) \left| \frac{\partial V(l)}{\partial \omega(m)} \right| + \lambda \tilde{r}(m, n) \right).$$

Whenever $\frac{72\sqrt{2}|g|}{\eta(1-e^{-2\nu})^d} < 1$ we have that

$$|g| \|\tilde{K}\|_{\infty,\infty} < 1 \tag{8.7}$$

where

$$\|\tilde{K}\|_{\infty,\infty} = \sup_l \sum_m \tilde{K}(l,m). \tag{8.8}$$

Considering the weight $W(n) := e^{\nu|m-n|}$ we let

$$\theta := \frac{3\|F\|_\infty}{2\pi} \sup_n \sum_l \frac{W(n)}{W(l)} \tilde{K}(n,l). \tag{8.9}$$

By the triangle inequality,

$$\theta \le \frac{3\|F\|_\infty}{2\pi} \sup_n \sum_l e^{\nu|n-l|} \tilde{K}(n,l)$$

$$\le \frac{72\sqrt{2}\|F\|_\infty}{\eta \left(1 - e^{-\nu}\right)^d}.$$

hence, whenever $\frac{72\sqrt{2}|g|}{\eta(1-e^{-\nu})^d} < 1$, we have that

$$|g|\theta < 1. \tag{8.10}$$

Moreover, with the choice

$$D_1 := \sum_n W(n)\tilde{r}(m,n) \tag{8.11}$$

we have

$$D_1 \le \frac{72\sqrt{2}\|F\|_\infty}{\eta \left(1 - e^{-\nu}\right)^d}. \tag{8.12}$$

After conditions (8.7), (8.10) and (8.12) have been verified, the general result [5, Theorem 9.2] applies, yielding

$$\sum_m e^{\nu|n-m|} \left|\frac{\partial V(n)}{\partial \omega(m)}\right| < \frac{\lambda D_1}{1 - g\theta} := C_1(d, \|F\|_\infty, \lambda, g, \eta, \nu). \tag{8.13}$$

Differentiating (8.4) with respect to $\omega(l)$,

$$\frac{\partial^2}{\partial\omega(m)\partial\omega(l)} \frac{1}{H-z} + g\left(\frac{\partial}{\partial\omega(l)} \frac{1}{H-z}\right) \frac{\partial V}{\partial\omega(m)} \frac{1}{H-z}$$

$$+ g\frac{1}{H-z} \frac{\partial V}{\partial\omega(m)} \left(\frac{\partial}{\partial\omega(l)} \frac{1}{H-z}\right) + g\frac{1}{H-z} \frac{\partial^2 V}{\partial\omega(m)\partial\omega(l)} \frac{1}{H-z}$$

$$= -\lambda\left(\frac{\partial}{\partial\omega(l)} \frac{1}{H-z}\right) P_m \frac{1}{H-z} - \lambda\frac{1}{H-z} P_m \left(\frac{\partial}{\partial\omega(l)} \frac{1}{H-z}\right)$$

Repeating the previous argument and using the established decay of $\frac{\partial V(n)}{\partial\omega(m)}$ we reach (8.2), finishing the proof. $\qquad\square$

Remark. Using the chain rule it is straightforward to check that

$$\frac{\partial V_{\text{eff}}(n)}{\partial U(m)} = \frac{\partial V_{\text{eff}}(n)}{\partial \omega(m)} - \frac{g}{\lambda} \sum_{l \in \mathbb{Z}^d} \frac{\partial V_{\text{eff}}(n)}{\partial \omega(l)} \frac{\partial V_{\text{eff}}(l)}{\partial U(m)}.$$

Thus, it follows from the above techniques that $\frac{\partial V_{\text{eff}}(n)}{\partial U(m)}$ and $\frac{\partial V_{\text{eff}}(n)}{\partial \omega(m)}$ have the same rate of decay.

Given a finite set $\Lambda \subset \mathbb{Z}^d$ and an arbitrary specification of the random variables ω, let us define $\mathcal{T} : \mathbb{R}^{|\Lambda|} \to \mathbb{R}^{|\Lambda|}$ by

$$(\mathcal{T}\omega)(n) = \omega(n) + \frac{g}{\lambda} V_{\text{eff}}(n). \tag{8.14}$$

Let $U(n) := (\mathcal{T}\omega)(n)$ be the new coordinates in the probability space. The bound (8.1) implies that if $\{\omega(n_k)\}_{k=1}^{|\Lambda|}$ and $\{\tilde{\omega}(n_k)\}_{k=1}^{|\Lambda|}$ are two configurations of the random variables in Λ we have that, for any $j \in \Lambda$,

$$\left| V_{\text{eff}}(j) \left(\omega(n_1), \ldots, \omega(n_{|\Lambda|}) \right) - V_{\text{eff}}(j) \left(\tilde{\omega}(n_1), \ldots, \tilde{\omega}(n_{|\Lambda|}) \right) \right|$$

$$\leq \sum_{k=1}^{|\Lambda|} \left| \frac{\partial V_{\text{eff}}(j)}{\partial \omega(n_k)} \right| |\omega(n_k) - \tilde{\omega}(n_k)|$$

$$\leq C_1 \sum_{k=1}^{|\Lambda|} e^{-\nu|j-n_k|} |\omega(n_k) - \tilde{\omega}(n_k)|.$$

Where both effective potentials are calculated with the same specification of the random variables $\{\omega(l)\}_{\{l \in \Lambda^c\}}$ outside Λ. Summing over j, we may conclude that the map below is Lipschitz with a Lipschitz constant independent of $|\Lambda|$:

$$\left(\omega(n_1), \ldots, \omega(n_{|\Lambda|}) \right) \mapsto \left(V_{\text{eff}}(n_1) \left(\omega(n_1), \ldots, \omega(n_{|\Lambda|}) \right), \ldots, \right.$$

$$\left. V_{\text{eff}}(n_{|\Lambda|}) \left(\omega(n_1), \ldots, \omega(n_{|\Lambda|}) \right) \right). \tag{8.15}$$

In particular, for $\frac{|g|}{\lambda}$ sufficiently small, $\mathcal{T}$ is a perturbation of the identity by a map with Lipschitz constant less than one. It follows from the contraction mapping theorem that $\mathcal{T}$ is a bijection, thus $\mathcal{T}^{-1} : \mathbb{R}^{|\Lambda|} \to \mathbb{R}^{|\Lambda|}$ is well defined.

Fix $n_0 \in \Lambda$ and denote by $U_\alpha = U + (\alpha - U(n_0)) \delta_{n_0}$ the new potential obtained from U by setting its value at n_0 equal to α. Let $\omega_\alpha(n) = \left(\mathcal{T}^{-1} U_\alpha \right)(n)$. The variables $\omega_\alpha(n)$ correspond to the change in $\omega(n)$ when a resampling argument is applied to the new probability space at the point n_0. Intuitively, the exponential decay guarantees that this change is not too large if n and n_0 are far away. This is the content of the lemma below.

Lemma 15. *For all $\alpha \in \mathbb{R}$ and $|g| < \lambda C_1^{-1}$, we have*

$$\sum_{nn \in \Lambda \setminus \{n_0\}} e^{\nu|n-n_0|} |\omega_\alpha(n) - \omega(n)| \leq \frac{C_1|g|}{\lambda} \frac{\left(|\alpha - U(n_0)| + 2\frac{|g| \|F\|_\infty}{\lambda} \right)}{\left(1 - \frac{|g|}{\lambda} C_1 \right)}.$$

where C_1 is the upper bound in Eq. (8.1).

Proof. Using the given definitions we obtain, for all n in Λ, that

$$|\omega(n) - \omega_\alpha(n)| \leq |\alpha - U(n_0)|\delta_{n_0}(n) + \frac{|g|}{\lambda}|V_{\text{eff}}(n, \omega) - V_{\text{eff}}(n, \omega_\alpha)|. \quad (8.16)$$

In particular, since we are only interested in the values $n \neq n_0$, the above simplifies to

$$\begin{aligned}
|\omega(n) - \omega_\alpha(n)| &\leq \frac{|g|}{\lambda}|V_{\text{eff}}(n, \omega) - V_{\text{eff}}(n, \omega_\alpha)| \\
&\leq \frac{|g|}{\lambda}\sum_{l \in \mathbb{Z}^d}\left|\frac{\partial V_{\text{eff}}(n, \hat{\omega}_\alpha)}{\partial \omega(l)}\right|\left|\omega_\alpha(l) - \omega(l)\right| \\
&\leq \frac{|g|}{\lambda}\left|\frac{\partial V_{\text{eff}}(n, \hat{\omega}_\alpha)}{\partial \omega(n_0)}\right|\left(|\alpha - U(n_0)| + 2\frac{|g|\|F\|_\infty}{\lambda}\right) \\
&\quad + \frac{|g|}{\lambda}\sum_{l \neq n_0}\left|\frac{\partial V_{\text{eff}}(n, \hat{\omega}_\alpha)}{\partial \omega(l)}\right|\left|\omega_\alpha(l) - \omega(l)\right|,
\end{aligned}$$

where we have used the mean value inequality and $\hat{\omega}_\alpha$ denotes some configuration with $\hat{\omega}_\alpha(l)$ in the interval connecting $\omega(l)$ to $\omega_\alpha(l)$. Let $W(n) = e^{\nu|n-n_0|}$. According to (8.1),

$$\sup_n \sum_l \frac{W(n)}{W(l)}\left|\frac{\partial V_{\text{eff}}(n, \hat{\omega}_\alpha)}{\partial \omega(l)}\right| \leq \sup_n \sum_l e^{\nu|n-l|}\left|\frac{\partial V_{\text{eff}}(n, \hat{\omega}_\alpha)}{\partial \omega(l)}\right|$$

$$\leq C_1.$$

Once again, the conditions of [5, Theorem 9.2] are satisfied for $|g| < \lambda C_1^{-1}$, therefore

$$\sum_{n \neq n_0} e^{\nu|n-n_0|}\left|\omega_\alpha(n) - \omega(n)\right| \leq \frac{C_1|g|}{\lambda}\frac{\left(|\alpha - U(n_0)| + 2\frac{|g|\|F\|_\infty}{\lambda}\right)}{\left(1 - \frac{|g|}{\lambda}C_1\right)}.$$

Since another application of the mean value inequality gives, after a possible correction to $\hat{\omega}_\alpha$, that

$$\left|\frac{\partial V_{\text{eff}}(n, \omega)}{\partial \omega(m)} - \frac{\partial V_{\text{eff}}(n, \omega_\alpha)}{\partial \omega(m)}\right| \leq \sum_{l \in \mathbb{Z}^d}\left|\frac{\partial^2 V_{\text{eff}}(n, \hat{\omega}_\alpha)}{\partial \omega(l)\partial \omega(m)}\right|\left|\omega(l) - \omega_\alpha(l)\right|,$$

we obtain, for any $\nu' \in (0, \nu)$,

$$\begin{aligned}
&\left|\frac{\partial V_{\text{eff}}(n, \omega)}{\partial \omega(m)} - \frac{\partial V_{\text{eff}}(n, \omega_\alpha)}{\partial \omega(m)}\right| \\
&\leq \frac{C_2|g|}{\lambda}\left(C_1\frac{\left(|\alpha - U(n_0)| + 2\frac{|g|\|F\|_\infty}{\lambda}\right)}{\left(1 - \frac{|g|}{\lambda}C_1\right)(1 - e^{\nu'-\nu})^d} + 2\|F\|_\infty\right)e^{-\nu'(|m-n|+|n-n_0|+|m-n_0|)},
\end{aligned}$$

where C_2 is the constant in (8.2).

In particular, letting $\nu' = \nu/2$, if $A = \frac{g}{\lambda}\left(\frac{\partial V_{\text{eff}}(n_i,\omega_\alpha)}{\partial \omega(n_j)}\right)_{|\Lambda|\times|\Lambda|}$ and $B = \frac{g}{\lambda}\left(\frac{\partial V_{\text{eff}}(n_i,\omega)}{\partial \omega(n_j)}\right)_{|\Lambda|\times|\Lambda|}$ we have

$$\sum_{(m,n)\in\Lambda\times\Lambda} |(A-B)_{m,n}| \leq \frac{C_2|g|^2}{\lambda^2}\left(C_1 \frac{\left(|\alpha - U(n_0)| + 2\frac{|g|\|F\|_\infty}{\lambda}\right)}{\left(1 - \frac{|g|}{\lambda}C_1\right)(1 - e^{-\nu/2})^{3d}} + \frac{2\|F\|_\infty}{(1 - e^{-\nu/2})^{2d}}\right).$$

$$(8.17)$$

We summarize the above observation as a lemma.

Lemma 16. *Let* $A = \frac{g}{\lambda}\left(\frac{\partial V_{\text{eff}}(n_i,\omega_\alpha)}{\partial \omega(n_j)}\right)_{|\Lambda|\times|\Lambda|}$ *and* $B = \frac{g}{\lambda}\left(\frac{\partial V_{\text{eff}}(n_i,\omega)}{\partial \omega(n_j)}\right)_{|\Lambda|\times|\Lambda|}$. *Whenever* $\frac{|g|}{\lambda}C_1 < 1$ *we have*

$$\sum_{(m,n)\in\Lambda\times\Lambda} |(A-B)_{m,n}| \leq |g|^2\left(C_3|\alpha - U(n_0)| + C_4\right).$$

$$(8.18)$$

Moreover, the constant C_3 *can be chosen independent of* λ *and* C_4 *is proportional to* $\frac{1}{\lambda}$.

Finally, we analyze how the effective potential varies with respect to disorder and interaction. This will be relevant to the Integrated Density of States regularity. More precisely

Lemma 17. *For a fixed* $\omega \in \Omega$

$$|V_{\lambda,g}(n) - V_{\lambda',g'}(n)|$$
$$\leq \frac{C_5(d,\|F\|_\infty,g,\eta,\nu,\omega)}{1 - gC_6(d,\|F\|_\infty,g,\eta,\nu)}|\lambda - \lambda'| + C_7(d,\|F\|_\infty,g,\eta,\nu)|g - g'|. \quad (8.19)$$

Note when $\lambda \neq \lambda'$ the bound depends on ω through the constant C_5.

Proof. Let $R_{\lambda,g}(z) = \frac{1}{H_0 + \lambda\omega + gV_{\lambda,g} - z}$ and $R_{\lambda',g'}(z) = \frac{1}{H_0 + \lambda'\omega + g'V_{\lambda',g'} - z}$ for $z = t + i\eta$. Similarly as in the above proofs, it is immediate to check that

$$R_{\lambda,g}(z) - R_{\lambda',g'}(z) = (\lambda - \lambda')R_{\lambda,g}(z)V_\omega R_{\lambda',g'}(z) + (g - g')R_{\lambda,g}(z)V_{\lambda',g'}R_{\lambda',g'}(z)$$
$$-gR_{\lambda,g}(z)\left(V_{\lambda,g} - V_{\lambda',g'}\right)R_{\lambda',g'}(z). \quad (8.20)$$

Replacing z by $\bar{z}$ and subtracting the resulting equations:

$$\left(R_{\lambda,g}(z) - R_{\lambda,g}(\bar{z})\right) - \left(R_{\lambda',g'}(z) - R_{\lambda',g'}(\bar{z})\right)$$
$$= \left(R_{\lambda,g}(z) - R_{\lambda,g}(\bar{z})\right)\left((\lambda - \lambda')V_\omega + (g - g')V_{\lambda',g'}\right)R_{\lambda',g'}(z)$$
$$+ R_{\lambda,g}(z)\left((\lambda - \lambda')V_\omega + (g - g')V_{\lambda',g'}\right)\left(R_{\lambda',g'}(z) - R_{\lambda',g'}(\bar{z})\right)$$
$$- gR_{\lambda,g}(z)\left(V_{\lambda,g} - V_{\lambda',g'}\right)\left(R_{\lambda',g'}(z) - R_{\lambda',g'}(\bar{z})\right).$$

Taking matrix elements, multiplying by $f(t)$, integrating with respect to t and taking absolute values we can read from the representation (8.3) that, denoting

$$K_{\lambda,g}(n,l) = |G_{\lambda,g}(n,l;z) - G_{\lambda,g}(n,l;\bar{z})|,$$
$$|V_{\lambda,g}(n) - V_{\lambda',g'}(n)|$$

$$\leq \frac{3\|F\|_\infty}{2\pi}|\lambda - \lambda'| \sum_{l \in \mathbb{Z}^d} |\omega(l)| \int_{-\infty}^{\infty} \Big(K_{\lambda,g}(n,l)K_{\lambda',g'}(l,n)$$

$$+|G_{\lambda,g}(n,l)|\tilde{K}_{\lambda',g'}(l,n)\Big)\, dt$$

$$+\frac{3\|F\|_\infty^2}{2\pi}|g - g'| \sum_{l \in \mathbb{Z}^d} \int_{-\infty}^{\infty} \Big(|G_{\lambda,g}(n,l)|K_{\lambda',g'}(l,n) + G_{\lambda',g'}(n,l)|K_{\lambda,g}(l,n)\Big)\, dt.$$

$$+g \sum_{l \in \mathbb{Z}^d} \int_{-\infty}^{\infty} |G_{\lambda,g}(n,l)||V_{\lambda,g}(l) - V_{\lambda',g'}(l)|K_{\lambda',g'}(l,n)\, dt. \tag{8.21}$$

Using Eq. 8.6 together with [5, Theorem 9.2] we conclude the proof. $\qquad\square$

8.1. Improvements. We will now improve upon the previous bounds. Specifically, we need robust bounds which also reflect the decay of the derivatives of $V_{\text{eff}}(n)$ when the local potential $\omega(n)$ is large. The improvements on this section will be important for a general fluctuation analysis in Sect. 9 and for localization in the one dimensional setting. Before stating the main result of the section we start with the following deterministic estimate, which incorporates ideas from [3, Lemma 3].

Lemma 18.

$$|G(m,l;t+i\eta)| \leq \sqrt{2}\langle l| \frac{1}{(H-t)^2 + \eta^2/2}|l\rangle^{1/2} e^{-\nu|m-l|} \tag{8.22}$$

Proof. To keep the notation simple, we set $t = 0$ without loss of generality. Let $H_f = e^f H e^{-f}$ where $f(n) = \nu \min\{|n - l|, R\}$ for a fixed $l \in \mathbb{Z}^d$ and $R > 0$. By choosing R sufficiently large, we may assume that $|m - l| < R$. We then have

$$e^{\nu|m-l|}G(m,l;i\eta) = \langle m|(H_f - i\eta)^{-1}|l\rangle.$$

We claim that

$$\|(H_f - i\eta)^{-1}(H^2 + \eta^2/2)^{1/2}\| \leq \sqrt{2}. \tag{8.23}$$

Indeed,

$$\|(H_f - i\eta)^{-1}(H^2 + \frac{\eta^2}{2})^{1/2}\|^2 = \|(H^2 + \frac{\eta^2}{2})^{1/2}(H_f^* + i\eta)^{-1}(H_f - i\eta)^{-1}(H^2 + \frac{\eta^2}{2})^{1/2}\|$$

$$= \|(H^2 + \frac{\eta^2}{2})^{1/2} \frac{1}{(H_f - i\eta)(H_f^* + i\eta)}(H^2 + \frac{\eta^2}{2})^{1/2}\|$$

where by [3, Eq D.9] (with f replaced by $-f$) we have

$$(H_f - i\eta)(H_f^* + i\eta) \geq \frac{1}{2}\left(H^2 + \frac{\eta^2}{2}\right)$$

showing the claim in (8.23). Lemma 18 will now follow from

$$|\langle m|(H_f - i\eta)^{-1}|l\rangle| \leq \|(H_f - i\eta)^{-1}(H^2 + \frac{\eta^2}{2})^{1/2}\|\,|(H^2 + \frac{\eta^2}{2})^{-1/2}\delta_l|$$

$$\leq \sqrt{2}\langle l|(H^2 + \frac{\eta^2}{2})^{-1}|l\rangle^{1/2}.$$

Lemma 19. *There exists $C_7(\lambda, \eta, d, g, \|F\|_\infty, v) > 0$ such that, for $m \neq n$,*

$$\max\{|\omega(n)|, |\omega(m)|\}\left|\frac{\partial V(n)}{\partial \omega(m)}\right| \leq C_7 e^{-2v|m-n|} \qquad (8.24)$$

and, for $n \neq n_0$,

$$\left|\omega(n)(\omega_\alpha(n) - \omega(n))\right|$$

$$\leq \frac{C_7|g|}{\lambda - |g|C_1}\left(|\alpha - U(n_0)| + 2\frac{|g|\|F\|_\infty}{\lambda} + \frac{1}{\left(1 - e^{-v}\right)^d}\right)e^{-v|n-n_0|}. \qquad (8.25)$$

Moreover, whenever $\frac{|g|}{\lambda}C_1 < 1$, C_7 can be chosen to be uniformly bounded as a function of the parameters λ and g.

Proof. Recall that $U(n) = \omega(n) + \frac{g}{\lambda}V_{\text{eff}}(n)$ denotes the "full" potential at site n. We split the proof in two cases.

(i) Case one: $U(n) \geq 0$.
Let us start by noting that Lemma 18 implies that for $n, l \in \mathbb{Z}^d$

$$\int_{-\infty}^\infty |G(n, l; t + i\eta)G(l, n; t + i\eta)|\, dt \leq \frac{2\sqrt{2}\pi}{\eta}e^{-2v|n-l|}. \qquad (8.26)$$

From the previous section we already know that

$$\frac{\partial V(n)}{\partial \omega(m)} = \int_{-\infty}^\infty \left(-g\sum_l r(n, l)\frac{\partial V(l)}{\partial \omega(m)} + \lambda r(m, n)\right)f(t)\, dt \qquad (8.27)$$

where $f(t) = F_+(t + i\eta) + F_-(t - i\eta) - D * F(t)$ and

$$r(m, n) = G(n, m; t + i\eta)G(m, n; t + i\eta) - G(n, m; t - i\eta)G(m, n; t - i\eta).$$

Observe that, for $z = t + i\eta$ and $n \neq m$,

$$\lambda|U(n)G(n, m; t + i\eta)G(m, n; t + i\eta)|$$

$$= \frac{\lambda|U(n)|}{|\lambda U(n) - z|}\sum_l |H_0(n, l)G(l, m; t + i\eta)G(m, n, t + i\eta)|$$

$$\leq \left(1 + \frac{|z|}{|\lambda U(n) - z|}\right)\sum_l |H_0(n, l)G(l, m; t + i\eta)G(m, n, t + i\eta)|$$

where we made use of the identity

$$(\lambda U(m) - z)G(n, m; z) = \delta_{mn} - \sum_l H_0(n, l)G(l, m; z). \qquad (8.28)$$

Note that if $U(n) \geq 0$ and $t = \operatorname{Re}z < 0$, then

$$\frac{|z|}{|\lambda U(n) - z|} \leq 1. \qquad (8.29)$$

Using the fact that $tf(t)$ goes to zero as $t \to \infty$ we conclude that

$$\int_{-\infty}^{\infty} |U(n)G(n,m;t+i\eta)G(m,n;t+i\eta)||f(t)|\, dt \leq \frac{C(\nu,\eta,\|F\|_{\infty})}{\lambda} e^{-2\nu|m-n|}. \tag{8.30}$$

Since a similar equation holds with m replaced by l, we can proceed as in the previous section and, using the exponential decay of $\frac{\partial V(n)}{\partial \omega(m)}$, conclude the proof.

(ii) Case two: $U(n) < 0$.

In this case, the argument given above must be modified to take into account that the inequality

$$\frac{|z|}{|\lambda U(n) - z|} \leq 1. \tag{8.31}$$

is satisfied when $t = \mathrm{Re} > 0$. In this case, the use of (8.27) would result in a problem as $tf(t)$ is unbounded as $t \to -\infty$. This can be addressed by observing that the Fermi-Dirac function $F(z)$ has the following symmetry

$$\frac{1}{1 + e^{\beta(z-\mu)}} = 1 - \frac{1}{1 + e^{\beta(-z+\mu)}}. \tag{8.32}$$

Hence we can make use of the representation (8.27) corresponding to

$$-\frac{1}{1 + e^{\beta(-z+\mu)}} := F_{\mu}^{*}(z) \tag{8.33}$$

since, for $m \neq n$, the constant term does not affect the calculation of $\frac{\partial V(n)}{\partial \omega(m)}$. Denoting by

$$f^{*}(t) = F_{+}^{*}(t+i\eta) + F_{-}^{*}(t-i\eta) - D * F^{*}(t) \tag{8.34}$$

we reach

$$\frac{\partial V(n)}{\partial \omega(m)} = \int_{-\infty}^{\infty} \left(-g \sum_{l} r(n,l) \frac{\partial V(l)}{\partial \omega(m)} + r(m,n) \right) f^{*}(t)\, dt \tag{8.35}$$

where now $tf^{*}(t) \to 0$ as $t \to -\infty$. Proceeding as in the first case the proof is finished, showing (8.24). Following the proof of Lemma 15 and using (8.24) we conclude (8.25)

$\square$

Our next result will be applied to the one dimensional context. Let $I = [0,l] \cap \mathbb{Z}$ and $J = [-r, l+r] \cap \mathbb{Z}$. Let $\{\omega(k)\}_{k \in J^c}$ be an arbitrary specification of the random variables outside J. Given a finite, arbitrary, list of real numbers $\{U(j)\}_{j \in J}$, there exists a list $\omega_J = \{\omega(j)\}_{j \in J}$ such that the effective potential $V_{\mathrm{eff}}(\omega) = V_{\mathrm{eff}}(\omega_J, \omega_{J^c})$, as defined in Sect. 6, satisfies $U(j) = \omega(j) + g V_{\mathrm{eff}}(\omega_J, \omega_{J^c})(j)$ for $j \in J$. Recall that in Sect. 8, specifically in (8.14), we have denoted $\mathcal{T}(\omega_J) = U_J$ and showed that $\mathcal{T} : \mathbb{R}^{|J|} \to \mathbb{R}^{|J|}$ is a bijection, so that the above reasoning is justified and $V_{\mathrm{eff}}(\omega_J, \omega_{J^c}) = V_{\mathrm{eff}}(\mathcal{T}^{-1}U_J, \omega_{J^c})$.

Similarly, let $K = ([-r-a, -r) \cup (l+r, l+r+b]) \cap \mathbb{Z}$ be a finite set with $a, b > 0$ and $d(K, I) = r+1$. Given two arbitrary lists of real numbers $\{U(j)\}_{j \in J}$ and $\{\tilde{U}(k)\}_{k \in K}$, let $V_{\mathrm{eff}}(\mathcal{T}^{-1}\left(U_J, \tilde{U}_K\right), \omega_{K^c})$ be the effective potential defined at specified values of

$\{\omega(l)\}_{l \in K^c}$ with $U(j) = \omega(j) + \frac{g}{\lambda} V_{\text{eff}}(\mathcal{T}^{-1}(U_J, \tilde{U}_K), \omega_{K^c})(j)$ for $j \in J$ and $\tilde{U}(k) = \omega(k) + \frac{g}{\lambda} V_{\text{eff}}(\mathcal{T}^{-1}(U_J, \tilde{U}_K), \omega_{K^c})(k)$ for $k \in K$. In what follows, it will be useful to think of $V_{\text{eff}}(\mathcal{T}^{-1} U_J, \omega_{J^c})$ as a function of U_J and ω_{J^c} instead of ω_J and ω_{J^c}. For that reason and for simplicity, we abuse notation writing

$$V_{\text{eff}}(U_J, \omega_{J^c}) = V_{\text{eff}}(\mathcal{T}^{-1} U_J, \omega_{J^c}) \tag{8.36}$$

and

$$V_{\text{eff}}(U_J, \tilde{U}_K, \omega_{K^c}) = V_{\text{eff}}(\mathcal{T}^{-1}(U_J, \tilde{U}_K), \omega_{K^c}). \tag{8.37}$$

The result below is a deterministic lemma which, intuitively speaking, states that, if $|m - n|$ is large, $V_{\text{eff}}(U_J, \omega_{J^c})(n)$ changes very little when the random variables $U(m)$ or $\omega(m)$ are replaced by different realizations $\tilde{U}(m)$ or $\tilde{\omega}(m)$.

Lemma 20 (Locality of the effective potential). *There exists $C = C(\eta, g, \lambda, v)$ finite and $v > 0$ such that:*

(a) Let $\{\omega(k)\}_{k \in J^c}$ and $\{\tilde{\omega}(k)\}_{k \in J^c}$ be two arbitrary realizations of the random variables outside J and $V_{\text{eff}}(U_J, \omega_{J^c})$, $V_{\text{eff}}(U_J, \tilde{\omega}_{J^c})$ be the effective potentials defined as above. Then, for each $n \in I$

$$\max\{1, |U(n)|\} |V_{\text{eff}}(U_J, \tilde{\omega}_{J^c})(n) - V_{\text{eff}}(U_J, \omega_{J^c})(n)| \leq C e^{-v\,\text{dist}(n, J^c)} \tag{8.38}$$

(b) For each $n \in I$, we have

$$\max\{1, |U(n)|\} |V_{\text{eff}}(U_J, U_K, \omega_{K^c})(n) - V_{\text{eff}}(U_J, \tilde{U}_K, \omega_{K^c})(n)| \leq C e^{-v\,\text{dist}(n, K)} \tag{8.39}$$

where $\text{dist}(n, X) = \min\{|n - x| : x \in X\}$.

Proof. The proofs of the above statements are similar. We give the details for (a) only. Let H be the operator associated to $(\{U(j)\}_{j \in J}, \{\omega(k)\}_{k \in J^c}, V_{\text{eff}}(U_J, \omega_{J^c}))$. More precisely, $H = H_0 + \omega + V_{\text{eff}}(U_J, \omega_{J^c})$ with $V_{\text{eff}}(U_J, \omega_{J^c})(n) = \langle n|F(H)|n\rangle$. Similarly, let $\tilde{H}$ be the operator associated to $(\{U(j)\}_{j \in J}, \{\tilde{\omega}(k)\}_{k \in J^c}, V_{\text{eff}}(U_J, \tilde{\omega}_{J^c}))$. Denote by $G(m, n; z)$ and $\tilde{G}(m, n; z)$ the respective Green's functions. Assume, firstly, that $|U(n)| \leq 1$. By the resolvent identity and since H and $\tilde{H}$ have the same potential within J, we know that

$$G(n, n; z) - \tilde{G}(n, n; z) = -\lambda \sum_{k \in J^c} G(n, k; z)$$
$$\left((\omega(k) - \tilde{\omega}(k)) + \frac{g}{\lambda}(V_{\text{eff}}(U_J, \omega_{J^c})(k) - V_{\text{eff}}(U_J, \tilde{\omega}_{J^c})(k))\right) \tilde{G}(k, n; z). \tag{8.40}$$

For a given $k \in J^c$, it follows from Lemma 18 that, whenever $\text{Re}\,z = t$ and $|\text{Im}\,z| = \eta$,

$$\int_{-\infty}^{\infty} |G(n, k; z)\tilde{G}(k, n; z)| \, dt \leq \frac{2\sqrt{2\pi}}{\eta} e^{-2v|n-k|}. \tag{8.41}$$

Thus, letting $f(t) = F_+(t + i\eta) + F_-(t - i\eta) - D * F(t)$, the quantity

$$\int_{-\infty}^{\infty} \sum_{k \in J^c} |G(n, k; z) \left(V_{\text{eff}}(U_J, \omega_{J^c})(k) - V_{\text{eff}}(U_J, \tilde{\omega}_{J^c})(k)\right) \tilde{G}(k, n; z)| |f(t)| \, dt \tag{8.42}$$

decays as $Ce^{-2vd(n,J^c)}$, due to the boundedness of the effective potentials. Along the lines of the proof of Lemma 19, making use of (8.28) applied to either H or $\hat{H}$, we may conclude that

$$\int_{-\infty}^{\infty} |\omega(k)| |G(n, k; z) \tilde{G}(k, n; z)| |f(t)| \, dt$$

and

$$\int_{-\infty}^{\infty} |\tilde{\omega}(k)| |G(n, k; z) \tilde{G}(k, n; z)| |f(t)| \, dt$$

both decay as $Ce^{-2v|n-k|}$ even when $\omega(k)$ and $\tilde{\omega}(k)$ are unbounded. In particular, the sums

$$\sum_{k \in J^c} \int_{-\infty}^{\infty} |\omega(k)| |G(n, k; z) \tilde{G}(k, n; z)| |f(t)| \, dt \tag{8.43}$$

and

$$\sum_{k \in J^c} \int_{-\infty}^{\infty} |\tilde{\omega}(k)| |G(n, k; z) \tilde{G}(k, n; z)| |f(t)| \, dt \tag{8.44}$$

both decay as $Ce^{-2vd(n,J^c)}$. The proof is finished by taking absolute values in (8.40), integrating against $|f(t)|$ and using the representation (8.3) for both $V_{\text{eff}}(U_J, \omega_{J^c})(n)$ and $V_{\text{eff}}(U_J, \tilde{\omega}_{J^c})(n)$.

When $|U(n)| > 1$, we need to improve upon the above again with ideas related to the proof of Lemma 19. Namely, multiplying (8.40) by $U(n)$ and taking absolute values, we must estimate three terms:

$$\int_{-\infty}^{\infty} |U(n)| |\omega(k)| |G(n, k; z) \tilde{G}(k, n; z)| |f(t)| \, dt,$$

$$\int_{-\infty}^{\infty} |U(n)| |\tilde{\omega}(k)| |G(n, k; z) \tilde{G}(k, n; z)| |f(t)| \, dt$$

and

$$\int_{-\infty}^{\infty} \sum_{k \in J^c} |U(n)| |G(n, k; z) \left(V_{\text{eff}}(U_J, \omega_{J^c})(k) - V_{\text{eff}}(U_J, \tilde{\omega}_{J^c})(k)\right) \tilde{G}(k, n; z)| |f(t)| \, dt. \tag{8.45}$$

The later decays as $Ce^{-2v|n-k|}$, again due to the identity (8.28) and Lemma 18. The first two are more challenging. It suffices to present the decay proof for one of them, say $\int_{-\infty}^{\infty} |U(n)| |\omega(k)| |G(n, k) \tilde{G}(k, n; z)| |f(t)| \, dt$. Since $\omega(k)$ and $U(k) := \omega(k) +$

$\frac{g}{\lambda} V_{\text{eff}}(U_J, \omega_{J^c})(k)$ differ by a bounded factor, we may verify the desired exponential decay for

$$\int_{-\infty}^{\infty} |U(n)||U(k)||G(n, k)\tilde{G}(k, n; z)||f(t)| \, dt.$$

$$\lambda G(n, k; z)U(k) = \overline{G(k, n; \overline{z})\lambda U(k)}$$

$$= \overline{\overline{z}G(k, n; \overline{z})} + \sum_{m \in \mathbb{Z}} \overline{G(n, m; \overline{z})} H_0(m, k)$$

$$= zG(n, k; z) + \sum_{m \in \mathbb{Z}} G(m, n; z) H_0(m, k)$$

$$= \lambda G(n, k; z)U(n) - \sum_{m' \in \mathbb{Z}} G(n, m'; z) H_0(m', k) + \sum_{m \in \mathbb{Z}} G(m, n; z) H_0(m, k).$$
(8.46)

Thus, we can essentially replace the terms $|G(n, k; z)U(k)\tilde{G}(k, n; z)|$ appearing in (8.46) by $|G(n, k; z)U(n)\tilde{G}(k, n; z)|$. Indeed, the terms $\sum_{k' \in \mathbb{Z}} G(n, k'; z) H_0(k', j)$ and $\sum_{k \in \mathbb{Z}} G(k, n; z) H_0(k, j)$ can be shown to decay as $Ce^{-\nu|n-j|}$ with the use of Lemma 18 and off-diagonal decay of H_0.

Thus, we are left estimating

$$|U(n)|^2 |G(n, k; z)\tilde{G}(k, n; z)|.$$
(8.47)

Since $k \neq n$, using the identities

$$G(n, k; z) = \frac{1}{(\lambda U(n) - z)} \sum_{l} G(n, l; z) H_0(l, k)$$

and

$$\tilde{G}(k, n; z) = \frac{1}{(\lambda U(n) - z)} \sum_{l'} \tilde{G}(l', n; z) H_0(l', k)$$

we reach

$$|U(n)|^2 |G(n, k; z)\tilde{G}(k, n; z)|$$

$$\leq \frac{|U(n)|^2}{|(\lambda U(n) - z)(\lambda U(n) - z)|} \sum_{l, l' \in \mathbb{Z}} |G(n, l; z)\tilde{G}(l', n; z) H_0(l', k) H_0(l, k)|.$$
(8.48)

The proof can be now be concluded with arguments identical to the ones in Lemma 19. Namely, if $U(n) > 0$ we choose to represent both $V_{\text{eff}}(U_J, \omega_{J^c})$ and $V_{\text{eff}}(U_J, \tilde{\omega}_{J^c})$ according to (6.2). Integrating (8.48) against $|f(t)|$, using Lemma 18 and the fact that $t^2 f(t)$ decays exponentially as $t \to \infty$ we conclude that

$$\int_{-\infty}^{\infty} |U(n)|^2 |G(n, k; z)\tilde{G}(k, n; z)||f(t)| \leq Ce^{-2\nu|n-k|}$$
(8.49)

when $U(n) > 0$. In case $U(n) < 0$ we make use of the symmetry in (8.32) to represent both $V_{\text{eff}}(U_J, \omega_{J^c})$ and $V_{\text{eff}}(U_J, \tilde{\omega}_{J^c})$ by integrating against $f^*(t)$, defined as in (8.34). Since the constant term cancels, Lemma 18 and the fact that $t^2 f^*(t)$ decays exponentially as $t \to -\infty$ imply (8.49) in this case. $\square$

Lemma 21. *For any $n \in I$ we have:*

(a)

$$\left| \frac{\partial V_{\text{eff}}(U_J, \omega_{J^c})(n)}{\partial \omega(m)} - \frac{\partial V_{\text{eff}}(U_J, \tilde{\omega}_{J^c})(n)}{\partial \omega(m)} \right| \leq Ce^{-\nu(|m-n|+r)}. \qquad (8.50)$$

(b)

$$\left| \frac{\partial V_{\text{eff}}(U_J, U_K, \omega_{K^c})(n)}{\partial \omega(m)} - \frac{\partial V_{\text{eff}}(U_J, \tilde{U}_K, \omega_{K^c})(n)}{\partial \omega(m)} \right| \leq Ce^{-\nu(|m-n|+r)}. \, (8.51)$$

Proof. The proof follows the same steps as in the previous results. In contrast to (a) in Lemma 20, to show the inequality (a) above we reduce the proof to analysing the difference between the quantities $r(m,n)$ and $\tilde{r}(m,n)$ given by

$$r(m,n) = G(n,m; t+i\eta)G(m,n; t+i\eta) - G(n,m; t-i\eta)G(m,n; t-i\eta)$$

and

$$\tilde{r}(m,n) = \tilde{G}(n,m; t+i\eta)\tilde{G}(m,n; t+i\eta) - \tilde{G}(n,m; t-i\eta)\tilde{G}(m,n; t-i\eta).$$

This is due to (8.4) and (8.5), which imply that $\frac{\partial V_{\text{eff}}(U_J,\omega_{J^c})(n)}{\partial \omega(m)}$ and $\frac{\partial V_{\text{eff}}(U_J,\tilde{\omega}_{J^c})(n)}{\partial \omega(m)}$ are obtained by integrating $r(m,n)$ and $\hat{r}(m,n)$, respectively, against $f(t)$.

We observe that

$$G(n,m; z)G(m,n; z) - \tilde{G}(n,m; z)\tilde{G}(m,n; z)$$
$$= G(n,m; z)\left(G(m,n; z) - \tilde{G}(m,n; z)\right) + \left(G(n,m; z) - \tilde{G}(n,m; z)\right)\tilde{G}(m,n; z).$$

Moreover,

$$G(m,n; z) - \tilde{G}(m,n; z) = -\lambda \sum_{k \in J^c} G(m,k; z)$$
$$\left((\omega(k) - \tilde{\omega}(k)) + \frac{g}{\lambda}\left(V_{\text{eff}}(U_J, \omega_{J^c})(k) - V_{\text{eff}}(U_J, \tilde{\omega}_{J^c})(k)\right)\right)\tilde{G}(k,n; z).$$
$$(8.52)$$

The proof is now finished using arguments identical to the proof of Lemma 14 and the improvement on Lemma 19. $\quad \square$

9. Proof of Lemma 5

In this section we show the existence of the effective density ρ_{eff}. Fix $\Lambda \subset \mathbb{Z}^d$ finite. Recall that we defined

$$U(n, \omega) := \omega(n) + \frac{g}{\lambda} V_{\text{eff}}(n, \omega). \qquad (9.1)$$

Until the end of this section we suppress the ω dependence on $U(n)$ and V_{eff}. Note that, for $m, n \in \Lambda$,

$$\frac{\partial U(m)}{\partial \omega(n)} = \delta_{mn} + \frac{g}{\lambda}\frac{\partial V_{\text{eff}}(m)}{\partial \omega(n)}. \qquad (9.2)$$

We have denoted the above change of variables by $\mathcal{T} : \mathbb{R}^{|\Lambda|} \to \mathbb{R}^{|\Lambda|}$, which reads

$$\mathcal{T}(\omega(n_1), \ldots, \omega(n_{|\Lambda|})) = (U(n_1), \ldots, U(n_{|\Lambda|})) \tag{9.3}$$

We can now compute the joint distribution of the $\{U(n)\}$. Using the fact that the random variables $\{\omega(n)\}_{n \in \mathbb{Z}^d}$ have a common density ρ we conclude that for all Borel sets $I_1, \ldots, I_N$ in $\mathbb{R}$:

$$\mathbb{P}\left(U(n_1) \in I_1, \ldots, U(n_{|\Lambda|}), \in I_{|\Lambda|}\right)$$

$$= \int_{\mathcal{T}^{-1}(I_1 \times \cdots \times I_{|\Lambda|})} \prod_{k=1}^{|\Lambda|} \rho(\omega(n_k)) \, d\omega(n_k)$$

$$= \int_{I_1 \times \cdots \times I_{|\Lambda|}} |\det J_{\mathcal{T}^{-1}}| \prod_{k=1}^{|\Lambda|} \rho\left(\mathcal{T}^{-1} U(n_k)\right) dU(n_1) \ldots dU(n_{|\Lambda|})$$

$$= \int_{I_1 \times \cdots \times I_{|\Lambda|}} \left|\det\left(I + \frac{g}{\lambda} \frac{\partial V_{\text{eff}}(n_i, \mathcal{T}^{-1} U)}{\partial U(n_j)}\right)\right| \prod_{k=1}^{|\Lambda|} \rho\left(U(n_k) - \frac{g}{\lambda} V_{\text{eff}}(n_k, \mathcal{T}^{-1} U)\right) dU(n_k).$$

Therefore the joint distribution of $\{U(n_k)\}_{k=1}^{|\Lambda|}$ is given by the measure

$$\left|\det\left(I + \frac{g}{\lambda} \frac{\partial V_{\text{eff}}(n_i, \mathcal{T}^{-1} U)}{\partial U(n_j)}\right)\right| \prod_{k=1}^{|\Lambda|} \rho\left(U(n_k) - \frac{g}{\lambda} V_{\text{eff}}(n_k, \mathcal{T}^{-1} U)\right) dU(n_1) \ldots dU(n_{|\Lambda|}). \tag{9.4}$$

It follows that for each $n_0 \in \Lambda$ the conditional expectation of $U(n_0)$ at specified values of $\{U(n)\}_{n \neq n_0}$ has a density given by

$$\rho_{n_0}^{\Lambda} = \frac{\left|\det\left(I + \frac{g}{\lambda} \frac{\partial V_{\text{eff}}(n_i, \mathcal{T}^{-1} U)}{\partial U(n_j)}\right)\right| \prod_{k=1}^{|\Lambda|} \rho\left(U(n_k) - \frac{g}{\lambda} V_{\text{eff}}(n_k, \mathcal{T}^{-1} U)\right)}{\int_{-\infty}^{\infty} \left|\det\left(I + \frac{g}{\lambda} \frac{\partial V_{\text{eff}}(n_i, \mathcal{T}^{-1} U^{\alpha})}{\partial U(n_j)}\right)\right| \prod_{k=1}^{|\Lambda|} \rho\left(U^{\alpha}(n_k) - \frac{g}{\lambda} V_{\text{eff}}(n_k, \mathcal{T}^{-1} U^{\alpha})\right) d\alpha} \tag{9.5}$$

Where $U^{\alpha}(n) := U(n) + (\alpha - U(n_0)) \delta_{n=n_0}$. This strategy naturally leads to the analysis of ratios of determinants. A sufficient condition for an upper bound to the right-hand side of (9.5) is to obtain positive constants $C = C_{\text{fluct}}(U(n_0))$ and $D = D(\alpha)$ which are independent of $|\Lambda|$ and such that the following estimates hold true

$$\frac{\left|\det\left(I + \frac{g}{\lambda} \frac{\partial V_{\text{eff}}(n_i, \mathcal{T}^{-1} U^{\alpha})}{\partial U(n_j)}\right)\right|}{\left|\det\left(I + \frac{g}{\lambda} \frac{\partial V_{\text{eff}}(n_i, \mathcal{T}^{-1} U)}{\partial U(n_j)}\right)\right|} \geq D(\alpha). \tag{9.6}$$

$$\int_{-\infty}^{\infty} D(\alpha) \rho\left(\alpha - \frac{g}{\lambda} V_{\text{eff}}(n_0, \mathcal{T}^{-1} U^{\alpha})\right) \prod_{n \in |\Lambda| \setminus \{n_0\}} \frac{\rho\left(U^{\alpha}(n) - \frac{g}{\lambda} V_{\text{eff}}(n, \mathcal{T}^{-1} U^{\alpha})\right)}{\rho\left(U(n) - \frac{g}{\lambda} V_{\text{eff}}(n, \mathcal{T}^{-1} U)\right)} d\alpha$$
$$\geq C_{\text{fluct}}(U(n_0)). \tag{9.7}$$

The bounds (9.7) and (9.6) readily imply that, setting $U(n_0) = u$

$$\rho_{n_0}^{\Lambda}(u) \leq \frac{\rho\left(u - \frac{g}{\lambda} V_{\text{eff}}(n_0, \mathcal{T}^{-1} U)\right)}{C_{\text{fluct}}(u)}. \tag{9.8}$$

Lemma 5 will follow from a precise control of the right-hand side of Eq. (9.8).

We now execute the strategy which was outlined above. The ratio of determinants can be controlled through the following bound, where $\|M\|_1$ denotes the trace norm of a matrix M.

Lemma 22. *Let A, B be square matrices with $I + B$ invertible. Then,*

$$\left|\frac{\det(I + A)}{\det(I + B)}\right| \leq e^{\|(A-B)(I+B)^{-1}\|_1}. \tag{9.9}$$

Proof. We make use of the elementary identities

$$\frac{\det(I + A)}{\det(I + B)} = \det(I + A)(I + B)^{-1} \tag{9.10}$$

and

$$(I + A)(1 + B)^{-1} = I + (A - B)(I + B)^{-1}. \tag{9.11}$$

The proof is now finished making use of the inequality

$$|\det(1 + M)| \leq e^{\|M\|_1}$$

which holds in the general setting of trace class operators, see [30, Lemma 3.3]   □

The triangle inequality for the trace norm implies the following.

Corollary 23. *Under the above conditions*

$$\left|\frac{\det(I + B)}{\det(I + A)}\right| \geq e^{-\sum_{m,n}|((A-B)(I+B)^{-1})_{mn}|} \tag{9.12}$$

Letting $A = \frac{g}{\lambda}\left(\frac{\partial V(n_i,\omega)}{\partial U(n_j)}\right)_{|\Lambda|\times|\Lambda|}$ and $B = \frac{g}{\lambda}\left(\frac{\partial V(n_i,\omega_\alpha)}{\partial U(n_j)}\right)_{|\Lambda|\times|\Lambda|}$ and using Lemma 14 we see that, for $|g| < \lambda C_1^{-1}$, $(1 + B)^{-1}$ has uniformly bounded operator norm. Using Lemma 16 and Corollary 23 we conclude that (9.6) holds with $D(\alpha) = e^{-|g|^2 C_3(|\alpha - U(n_0)| + C_4)}$.

We now check that Eq. (9.7) holds when ρ satisfies the fluctuation bound (2.4). We divide the proof in two cases:

(I) Suppose that $c_2(\rho) > 0$.
Let $c_\rho = \max\{c_1(\rho), c_2(\rho)\}$. The left-hand side of (9.7) is bounded from below by

$$\int_{-\infty}^{\infty} D(\alpha)\rho\left(\alpha - \frac{g}{\lambda}V_{\text{eff}}(n_0, \mathcal{T}^{-1}U^\alpha)\right) \prod_{n\in\Lambda\setminus\{n_0\}} e^{-c_\rho|\omega(n)-\omega_\alpha(n)|(1+|\omega(n)|+|\omega_\alpha(n)|)} \, d\alpha \tag{9.13}$$

which equals

$$\int_{-\infty}^{\infty} D(\alpha)\rho\left(\alpha - \frac{g}{\lambda}V_{\text{eff}}(n_0, \mathcal{T}^{-1}U^\alpha)\right) e^{\sum_{n\in\Lambda\setminus\{n_0\}} -c_\rho|\omega(n)-\omega_\alpha(n)|(1+|\omega(n)|+|\omega_\alpha(n)|)} \, d\alpha. \tag{9.14}$$

Due to the triangle inequality and Lemmas 15 and 19, we conclude that there is a positive constant $C = C(d, \|F\|_\infty, g, \eta, \nu)$ with $\lim_{g\to 0} C(d, \|F\|_\infty, g, \eta, \nu) < \infty$ such that for $n \neq n_0$

$$
\begin{aligned}
&- c_\rho |\omega(n) - \omega_\alpha(n)| \left(1 + |\omega(n)| + |\omega_\alpha(n)|\right) \\
&\geq -c_\rho |\omega(n) - \omega_\alpha(n)| \left(1 + 2|\omega(n)| + |\omega_\alpha(n) - \omega(n)|\right) \\
&\geq -\frac{|g|}{\lambda} c_\rho e^{-\nu|n-n_0|} \left(C^2 |\alpha - U(n_0)|^2 + 2C|\alpha - U(n_0)|\right).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&- c_\rho \sum_{n\in\Lambda\setminus\{n_0\}} |\omega(n) - \omega_\alpha(n)| \left(1 + |\omega(n)| + |\omega_\alpha(n)|\right) \\
&\geq -\frac{|g|}{\lambda} \frac{2c_\rho}{\left(1 - e^{-\nu}\right)^d} \left(C^2 |\alpha - U(n_0)|^2 + 2C|\alpha - U(n_0)|\right)).
\end{aligned}
$$

Thus, by choosing $\frac{|g|}{\lambda}$ sufficiently small so that $\frac{|g|}{\lambda} \frac{4c_\rho}{(1-e^{-\nu})^d} C^2 < \bar{c}_\rho$ and using the assumption (A_4) we obtain

$$0 < \int_{-\infty}^{\infty} D(\alpha) \frac{\rho\left(\alpha - \frac{g}{\lambda} V_{\text{eff}}(n_0, \mathcal{T}^{-1} U^\alpha)\right)}{\rho\left(U(n_0) - \frac{g}{\lambda} V_{\text{eff}}(n_0, \mathcal{T}^{-1} U)\right)} \tag{9.15}$$

$$\times \prod_{n\in\Lambda\setminus\{n_0\}} e^{-c_\rho|\omega(n)-\omega_\alpha(n)|(1+|\omega(n)|+|\omega_\alpha(n)|)} \, d\alpha < \infty. \tag{9.16}$$

this, together with (9.8), verifies Lemma 5 when $c_2(\rho) > 0$. If ρ satisfies the assumption (A_5) the above argument yields $\rho_{n_0}^\Lambda(u) \in L^1\left(\mathbb{R}, |x|^\varepsilon dx\right)$.

(II) Assume that $c_2(\rho) = 0$:
Similarly to the above argument, the left-hand side of (9.7) is bounded from below by

$$\int_{-\infty}^{\infty} D(\alpha) \frac{\rho\left(\alpha - \frac{g}{\lambda} V_{\text{eff}}(n_0, \mathcal{T}^{-1} U^\alpha)\right)}{\rho\left(U(n_0) - \frac{g}{\lambda} V_{\text{eff}}(n_0, \mathcal{T}^{-1} U)\right)} e^{-\frac{|g|}{\lambda} \frac{c_1(\rho)}{(1-e^{-\nu})^d} C|\alpha-U(n_0)|}. \tag{9.17}$$

Where, from (2.4)

$$\frac{\rho\left(\alpha - \frac{g}{\lambda} V_{\text{eff}}(n_0, \mathcal{T}^{-1} U^\alpha)\right)}{\rho\left(U(n_0) - \frac{g}{\lambda} V_{\text{eff}}(n_0, \mathcal{T}^{-1} U)\right)} \geq e^{-c_1(\rho)\left(|\alpha-U(n_0)|+2\frac{|g|}{\lambda}\right)}. \tag{9.18}$$

Again, choosing $|g|$ sufficiently small we conclude that

$$0 < \int_{-\infty}^{\infty} D(\alpha) \frac{\rho\left(\alpha - \frac{g}{\lambda} V_{\text{eff}}(n_0, \mathcal{T}^{-1} U^\alpha)\right)}{\rho\left(U(n_0) - \frac{g}{\lambda} V_{\text{eff}}(n_0, \mathcal{T}^{-1} U)\right)} \prod_{n\in\Lambda\setminus\{n_0\}} e^{-c_1(\rho)|\omega(n)-\omega_\alpha(n)|} \, d\alpha < \infty. \tag{9.19}$$

finishing the proof.

10. The Hartree Approximation for the Hubbard Model

Let us now adapt the techniques from the previous sections to the Hubbard model. Recall that H_{Hub} is defined as

$$\begin{pmatrix} H_\uparrow(\omega) & 0 \\ 0 & H_\downarrow(\omega) \end{pmatrix} := \begin{pmatrix} H_0 + \lambda V_\omega + g V_\uparrow(\omega) & 0 \\ 0 & H_0 + \lambda V_\omega + g V_\downarrow(\omega) \end{pmatrix}$$

acting on $\ell^2\left(\mathbb{Z}^d\right) \oplus \ell^2\left(\mathbb{Z}^d\right)$. The operators H_0 and V_ω are defined as before, i.e; $H_0 + \lambda V_\omega$ is the standard Anderson model acting on $\ell^2\left(\mathbb{Z}^d\right)$. The effective potentials are defined as

$$\begin{pmatrix} V_\uparrow(\omega)(n) \\ V_\downarrow(\omega)(n) \end{pmatrix} = \begin{pmatrix} \langle n|F(H_\downarrow)|n\rangle \\ \langle n|F(H_\uparrow)|n\rangle \end{pmatrix}. \tag{10.1}$$

Mathematically, the treatment of the above model is very similar to the proof of Theorem 2 above, therefore some details are skipped and we just indicate the required modifications.

10.1. Existence of the effective potential. Let $\Phi(X, Y) : \ell^\infty\left(\mathbb{Z}^d\right) \oplus \ell^\infty\left(\mathbb{Z}^d\right) \to \ell^\infty\left(\mathbb{Z}^d\right) \oplus \ell^\infty\left(\mathbb{Z}^d\right)$ be given by

$$\Phi(X, Y)(m, n) := \left(\langle n|F(H_0 + V_\omega + gY)|n\rangle, \langle m|F(H_0 + V_\omega + gX)|m\rangle\right).$$

using Proposition 12, we immediately reach

$$\|\Phi(X_1, Y_1) - \Phi(X_2, Y_2)\|_{\ell^\infty(\mathbb{Z}^d) \oplus \ell^\infty(\mathbb{Z}^d)}$$

$$\leq |g| \frac{72\sqrt{2}}{\eta\left(1 - e^{v'-v}\right)^d} \|F\|_\infty \left(\|X_1 - X_2\|_{\ell^\infty(\mathbb{Z}^d)} + \|Y_1 - Y_2\|_{\ell^\infty(\mathbb{Z}^d)}\right). \tag{10.2}$$

Therefore, if $|g| \frac{72\sqrt{2}}{\eta\left(1-e^{v'-v}\right)^d} \|F\|_\infty < 1$ we conclude Φ has a unique fixed point $V_{\text{eff}} = (V_\uparrow, V_\downarrow)$ belonging to $\ell^\infty\left(\mathbb{Z}^d\right) \oplus \ell^\infty\left(\mathbb{Z}^d\right)$.

10.2. Regularity of the effective potential. Fix $\Lambda \subset \mathbb{Z}^d$ finite and define functions $\xi : (\ell^\infty(\Lambda) \oplus \ell^\infty(\Lambda)) \times \mathbb{R}^n \to \ell^\infty(\Lambda) \oplus \ell^\infty(\Lambda)$ through

$$\xi_\uparrow(V, \omega)(j) = V_\uparrow(j) - \langle j|F(H_0 + \lambda\omega + g V_\downarrow)|j\rangle. \tag{10.3}$$
$$\xi_\downarrow(V, \omega)(j) = V_\downarrow(j) - \langle j|F(H_0 + \lambda\omega + g V_\uparrow)|j\rangle. \tag{10.4}$$

Our goal is to conclude $V^\uparrow, V^\downarrow$ are smooth functions of an arbitrary, but finite, list $(\omega(1), \ldots, \omega(n))$. Again, this can be done via implicit function theorem once we check that the derivative

$$\frac{\partial\xi(\omega, V)(j)}{\partial V(l)} = \delta_{jl} - \frac{\partial\langle j|F(H_0 + \lambda\omega + gV)|j\rangle}{\partial V(l)}. \tag{10.5}$$

is non-singular. Using Lemma 12, we have that for $\sharp \in \{\uparrow, \downarrow\}$

$$\left|\frac{\partial \langle j | F(H_0 + \lambda\omega + g V_\sharp) | j \rangle}{\partial V(l)}\right| \leq |g| \frac{72\sqrt{2} e^{-2\nu|j-l|}}{\eta} \|F\|_\infty. \tag{10.6}$$

In particular, whenever $|g| \frac{144\sqrt{2}\|F\|_\infty}{\eta(1-e^{-2\nu})^d} < 1$ we have that the operator $D\xi(\omega, .) :$ $\ell^\infty(\Lambda) \oplus \ell^\infty(\Lambda) \to \ell^\infty(\Lambda) \oplus \ell^\infty(\Lambda)$ has an inverse. From the implicit function theorem it follows that V is a smooth function of $(\omega(1), \ldots, \omega(n))$ for $n = |\Lambda|$.

10.3. Decay estimates.
The decay rate in the case of the Hubbard model is dictated by

$$\left|\frac{\partial V_\uparrow(n)}{\partial \omega(m)}\right| \leq 3|g|\|F\|_\infty \sum_l \tilde{K}_\downarrow(l, m) \left|\frac{\partial V_\downarrow(l)}{\partial \omega(m)}\right| + \tilde{r}_\downarrow(n). \tag{10.7}$$

$$\left|\frac{\partial V_\downarrow(n)}{\partial \omega(m)}\right| \leq 3|g|\|F\|_\infty \sum_l \tilde{K}_\uparrow(l, m) \left|\frac{\partial V_\uparrow(l)}{\partial \omega(m)}\right| + \tilde{r}_\uparrow(n). \tag{10.8}$$

where, for $\sharp \in \{\uparrow, \downarrow\}$

$$\tilde{G}_\sharp(l, m) := G_\sharp(l, n; t + i\eta)G_\sharp(n, l; t + i\eta) - G_\sharp(l, n; t - i\eta)G_\sharp(n, l; t - i\eta).$$
$$r_\sharp(m, n) := G_\sharp(n, m; t + i\eta)G_\sharp(m, n; t + i\eta) - G_\sharp(n, m; t - i\eta)G_\sharp(m, n; t - i\eta).$$
$$\tilde{K}_\sharp(l, m) := \int_{-\infty}^\infty |\tilde{G}_\sharp(l, m)| \, dt.$$
$$\tilde{r}_\sharp(n) := \int_{-\infty}^\infty |r_\sharp(n)| \, dt.$$

In particular,

$$\left|\frac{\partial V_\uparrow(n)}{\partial \omega(m)}\right| + \left|\frac{\partial V_\downarrow(n)}{\partial \omega(m)}\right| \leq 3|g|\|F\|_\infty \sum_l \left(\tilde{K}_\uparrow(l, m) + \tilde{K}_\downarrow(l, m)\right)$$
$$\times \left(\left|\frac{\partial V_\uparrow(l)}{\partial \omega(m)}\right| + \left|\frac{\partial V_\downarrow(l)}{\partial \omega(m)}\right|\right)$$
$$+ \left(\tilde{r}_\uparrow(n, m) + \tilde{r}_\downarrow(n, m)\right). \tag{10.9}$$

The analysis from the previous sections applies and we obtain Lemmas 14, 15, 16, 19, and 20 and 21 with $|.|$ being replaced by the matrix norm $|M| = |M_{11}| + |M_{21}|$ for $M = \begin{pmatrix} M_{11} \\ M_{21} \end{pmatrix}$. The effective potential and its derivatives are to be interpreted as follows:

$$V_{\text{eff}}(n) = \begin{pmatrix} V_{\text{eff}}^\uparrow(n) \\ V_{\text{eff}}^\downarrow(n) \end{pmatrix}, \quad \frac{\partial V_{\text{eff}}(n)}{\partial \omega(m)} = \begin{pmatrix} \frac{\partial V_{\text{eff}}^\uparrow(n)}{\partial \omega(m)} \\ \frac{\partial V_{\text{eff}}^\downarrow(n)}{\partial \omega(m)} \end{pmatrix} \quad \text{and} \quad \frac{\partial^2 V_{\text{eff}}(n)}{\partial \omega(m)\partial \omega(l)} = \begin{pmatrix} \frac{\partial^2 V_{\text{eff}}^\uparrow(n)}{\partial \omega(m)\omega(l)} \\ \frac{\partial^2 V_{\text{eff}}^\downarrow(n)}{\partial \omega(m)\omega(l)} \end{pmatrix}.$$

11. One Dimensional Aspects: Proof of Theorem 1

In this section we will prove Theorem 1. We let $H_+ = H_{[0,\infty)\cap\mathbb{Z}}$ and denote by $G^+(m, n; z)$ the Green's function of H^+. Recall the definition of the Lyapunov exponent:

$$\mathcal{L}(z) = -\mathbb{E}\left(\ln|G^+(0, 0; z)|\right) \tag{11.1}$$

$$\mathcal{L}_{\text{And}}(z) = -\mathbb{E}\left(\ln|G^+_{\text{And}}(0, 0; z)|\right). \tag{11.2}$$

Recall in this case $H_0 = -\Delta$ hence, we define H_{Hub} acting on $\left(\ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})\right)$ by

$$H_{\text{Hub}} = \begin{pmatrix} H_\uparrow(\omega) & 0 \\ 0 & H_\downarrow(\omega) \end{pmatrix} \tag{11.3}$$

where, denoting by H_{And} the standard Anderson model $-\Delta + V_\omega$ on $\ell^2(\mathbb{Z})$,

$$\begin{pmatrix} H_\uparrow(\omega) & 0 \\ 0 & H_\downarrow(\omega) \end{pmatrix} := \begin{pmatrix} H_{\text{And}} + g V_\uparrow(\omega) & 0 \\ 0 & H_{\text{And}} + g V_\downarrow(\omega) \end{pmatrix}. \tag{11.4}$$

The effective potentials are defined as (2.8). In the theorem below, we will use an abbreviation and $\mathcal{L}(z)$ will refer to the Lyapunov exponent of either $H_\uparrow$ or $H_\downarrow$ whereas $\mathcal{L}_{\text{And}}(z)$ will denote the Lyapunov exponent of the Anderson model on $\ell^2(\mathbb{Z})$.

11.1. Proof of Lemma 10. Since $a_{n+m+r} \leq a_n + a_m + C$, the related sequence $b_n := a_n + C$ satisfies $b_{n+m+r} \leq b_n + b_m$ and $\frac{a_n}{n}$ converges to $\inf_n \frac{a_n+C}{n}$ if and only if $\frac{b_n}{n}$ converges to $\inf_n \frac{b_n}{n}$. Thus, without loss of generality, we may set $C = 0$ for simplicity.

Given integers L and ℓ with $L >> \ell$, our goal is to bound $\frac{a_L}{L}$ from above in terms of $\frac{a_\ell}{\ell}$. As a initial step, observe that by (5.17) we have

$$a_L \leq a_{L-\lfloor \delta \log L \rfloor - \ell} + a_\ell.$$

Iterating the above procedure $k + 1$ times for

$$k = k_{\ell,L} := \lfloor \frac{L - 2\ell - \delta \log L}{\delta \log L + \ell} \rfloor \tag{11.5}$$

we obtain

$$a_L \leq (k + 2) a_\ell$$

In the above iteration we have made use of the fact that in the Assumption (5.17) the remainder r can be adjusted as long as it satisfies the inequality given there. Thus,

$$\frac{a_L}{L} \leq \frac{\ell(k+2)}{L} \frac{a_\ell}{\ell} \tag{11.6}$$

Before proceeding with the proof, a few remarks are in order. Firstly, nothing is achieved by holding ℓ fixed and letting $L \to \infty$ directly on Eq. (11.6) since this only yields the upper bound of zero. A second attempt would be showing that letting $\ell \to \infty$ (hence, $L \to \infty$ as well) implies that the ratio $\frac{k\ell}{L}$ converges to one. However, as

$$q_{\ell,L} - \frac{\ell}{L} \leq \frac{k\ell}{L} \leq q_{\ell,L} \tag{11.7}$$

for the choice

$$q_{\ell,L} = \frac{1 - 2\frac{\ell}{L} - \delta\frac{\log L}{L}}{1 + \frac{\delta \log L}{\ell}} \tag{11.8}$$

we see that $\frac{k\ell}{L}$ converges to one as $\ell \to \infty$ only along a subsequence where

$$\frac{\ell}{L} \to 0 \quad \text{and} \quad \frac{\log L}{\ell} \to 0. \tag{11.9}$$

Taking this into account, let $\varepsilon > 0$ be given and ℓ_1 be an initial scale to be determined. Let $L >> \ell_1$ be a positive integer to be interpreted as a larger scale. Iterating Eq. (11.6) throughout a sequence of scales

$$\ell_1 < \ell_2 < \cdots < \ell_{N_L} \leq L < \ell_{N_L+1} < \cdots \tag{11.10}$$

satisfying, for some $p > 1$,

$$p \log \left(\ell_j \right) \leq \log \ell_{j+1} < p^2 \log \left(\ell_j \right). \tag{11.11}$$

and

$$\sum_{j=1}^{\infty} \frac{\ell_j}{\ell_{j+1}} < \infty \tag{11.12}$$

we reach, for $q_{\ell,L}$ defined in (11.8),

$$\frac{a_L}{L} \leq \left(q_{\ell_{N_L},L} + \frac{2\ell_{N_L}}{L} \right) \prod_{j=1}^{N_L-1} \left(q_{\ell_j,\ell_{j+1}} + \frac{2\ell_j}{\ell_{j+1}} \right) \frac{a_{\ell_1}}{\ell_1}. \tag{11.13}$$

Since $q_{\ell_j,\ell_{j+1}} \to 1$ as $j \to \infty$ Due to (11.12), we conclude that the value of ℓ_1 can be chosen(independently of L) so that

$$\sum_{j=1}^{N_L-1} \log \left(q_{\ell_j,\ell_{j+1}} + \frac{2\ell_j}{\ell_{j+1}} \right) + \log \left(q_{\ell_{N_L},L} + \frac{2\ell_{N_L}}{L} \right) < \varepsilon. \tag{11.14}$$

Thus

$$\frac{a_L}{L} \leq e^{\varepsilon} \frac{a_{\ell_1}}{\ell_1}. \tag{11.15}$$

Moreover, the above conclusion holds for any integer ℓ_1 sufficiently large, as long and $L >> \ell_1$. In particular, we can also require that

$$\frac{a_{\ell_1}}{\ell_1} \leq \inf_n \frac{a_n}{n} + \varepsilon. \tag{11.16}$$

Combining Eqs. (11.15) and (11.16) the proof is finished letting $\varepsilon \to 0$.

11.2. Proof of Lemma 11. Before proceeding with the proof, let us note that use will be made of Hardy notation: C will be used for various constants depending on $(\eta, g, \|F\|_\infty)$.

Recall that $U(j) = \omega(j) + g V_{\text{eff}}(n)$, where the effective potential is defined in $\mathbb{Z}$ according to Proposition 13. Without loss of generality, we may assume that $m \geq n$, in which case we let $r = \lfloor \delta \log m \rfloor$ with $\delta > 0$ to be determined. We will prove the result with r replaced by $2r$, for convenience. In order to find an upper bound for $\mathbb{E}\left(|\hat{G}(0, n + m + 2r; z)|^s\right)$ we first estimate the conditional expectation below

$$\mathbb{E}\left(|\hat{G}(0, n + m + 2r; z)|^s \ |\omega(j) \ , \ j \in \mathbb{Z} \cap [-2r, n + m + 4r]^c\right). \quad (11.17)$$

Note that $\hat{G}(0, n + m + 2r; z)$ is a function of $U(0), \ldots, U(n + m + 2r)$. Let $I = [-2r, n + m + 4r]$ and change variables in $\mathbb{R}^{n+m+6r}$ according to

$$\mathcal{T}_I\left(\omega(-2r), \ldots, \omega(n + m + 4r)\right) := \left(U(-2r), \ldots, U(n + m + 4r)\right). \quad (11.18)$$

The reason for changing the variables in a set slightly bigger than $\{0, 1, \ldots, n + m + 2r\}$ will be clear from the remainder of the proof. We may compute the conditional expectation in (11.17) by integrating $|\hat{G}(0, n + m + 2r; z)|^s$ with respect to the measure

$$\prod_{k \in I} \rho\left(U(k) - g V_{\text{eff}}(k)\right) \mathcal{J}_{I \times I} \, dU(-2r) \ldots dU(n + m + 4r)$$
$$:= \Pi(U_I, \omega_{I^c}) \, dU(-2r) \ldots dU(n + m + 4r). \quad (11.19)$$

where

$$\mathcal{J}_{I \times I} := \det\left(I + g \frac{\partial V_{\text{eff}}}{\partial U}\right)_{I \times I} \quad (11.20)$$

under the convention

$$\det\left(I + g \frac{\partial V_{\text{eff}}}{\partial U}\right)_{I \times I} = \det\left(\delta_{kl} + g \frac{\partial V_{\text{eff}}(k)}{\partial U(l)}\right) \quad k \in I \cap \mathbb{Z}, \ l \in I \cap \mathbb{Z} \quad (11.21)$$

which will be used throughout the proof. We remark that, by definition, the above effective potential is calculated at specified values of $\omega(j), \ j \in \mathbb{Z} \cap I^c$ so, in the notation of Lemmas 20 and 21 , $V_{\text{eff}} = V_{\text{eff}}(U_I, \omega_{I^c})$.

When estimating (11.17) from above, the first step is to integrate out the random variables $U(n+1)$ and $U(n+2r)$ with the help of the local *a-priori* bound from Lemma 4 and the identity

$$\hat{G}(0, n + m + 2r; z) = \hat{G}(0, n; z) G_{[0, n+m+2r]}(n + 1, n + 2r) \hat{G}(n + 2r + 1, n + m + 2r; z).$$

We may conclude that

$$\mathbb{E}\left(|\hat{G}(0, n + m + 2r; z)|^s \ |\omega(j) \ , \ j \in \mathbb{Z} \cap I^c\right)$$
$$\leq C_{\text{AP}} \mathbb{E}\left(|\hat{G}(0, n; z)|^s |\hat{G}(n + 2r + 1, n + 2r + m; z)|^s \ |\omega(j) \ , \ j \in \mathbb{Z} \cap I^c\right) \quad (11.22)$$

where use was made of the fact that the variables $U(n + 1)$ and $U(n + 2r)$ do not enter the product $|\hat{G}(0, n; z)|^s |\hat{G}(n + 2r + 1, n + 2r + m; z)|^s$. We will check below that the expectation in (11.22) can be decoupled as follows:

$$
\mathbb{E}\left(|\hat{G}(0, n; z)|^s|\hat{G}(n + 2r + 1, n + 2r + m; z)|^s \ |\omega(j) \ , j \in \mathbb{Z} \cap I^c\right)
$$
$$
\leq C^{2r+1}\mathbb{E}\left(|\hat{G}(0, n; z)|^s| \ |\omega(j) \ , j \in \mathbb{Z} \cap I^c\right) \tag{11.23}
$$
$$
\mathbb{E}\left(\hat{G}(n + 2r + 1, n + 2r + m; z)|^s \ |\omega(j) \ , j \in \mathbb{Z} \cap I^c\right).
$$

The above will be a corollary of the deterministic inequality between densities defined by (11.19)

$$
\Pi(U_I, \omega_{I^c}) \leq C^{2r+1}\Pi(U_{I_1}, \tilde{U}_{I_2}, \omega_{I^c})\Pi(\tilde{U}_{I_1}, U_{I_2}, \omega_{I^c}) \tag{11.24}
$$

where $I_1 = [-2r, n + r] \cap \mathbb{Z}$ and $I_2 = [n + r + 1, n + 4r + m] \cap \mathbb{Z}$ and $\{\tilde{U}(j)\}_{j \in I}$ is an arbitrary list of numbers, more details on this inequality will be given below. Moreover, we will prove that, given an arbitrary realization $\{\tilde{\omega}(j)\}_{j \in I^c}$, the following holds true:

$$
\mathbb{E}\left(|\hat{G}(0, n; z)|^s| \ |\omega(j) \ , j \in \mathbb{Z} \cap I^c\right) \leq C^{2r+1}\mathbb{E}\left(|\hat{G}(0, n; z)|^s| \ |\tilde{\omega}(j) \ , j \in \mathbb{Z} \cap I^c\right). \tag{11.25}
$$

where the above inequality will also follow from a deterministic bound on the densities, namely:

$$
\Pi(U_I, \omega_{I^c}) \leq C^{2r+1}\Pi(U_I, \tilde{\omega}_{I^c}). \tag{11.26}
$$

Lemma 11 follows from the bounds (11.22), (11.23) (combined with translation invariance) and (11.25) by integrating over $\omega_{I^c} = \{\omega(j)\}_{j \in I^c}$ and $\tilde{\omega}_{I^c} = \{\tilde{\omega}(j)\}_{j \in I^c}$ once we choose $\tilde{\omega}_{I^c}$ to be independent of ω_{I^c}, which can be done since in (11.25) $\tilde{\omega}_{I^c}$ is an arbitrary list of real numbers. Indeed, we show next that the decoupling estimates (11.23) and (11.25) follow from the deterministic estimates of Lemmas 20 and 21.

Let us start by proving (11.23). To estimate that conditional expectation, we recall the change of variables (11.18) and control the Jacobian determinant in (11.20) and product of densities in (11.19) as follows. Let

$$
I_1 = [-2r, n + r] \cap \mathbb{Z}, \ I_2 = [n + r + 1, n + m + 4r] \cap \mathbb{Z} \tag{11.27}
$$

so that $I = I_1 \cup I_2$. Split each of the intervals I_1 and I_2 according to $I_1 = I_{1,1} \cup I_{1,2}$ and $I_2 = I_{2,1} \cup I_{2,2}$ where

$$
I_{1,1} = [-2r, n] \cap \mathbb{Z}, \ I_{1,2} = [n + 1, n + r],
$$
$$
I_{2,1} = [n + r + 1, n + 2r] \cap \mathbb{Z}, \ I_{2,2} = [n + 2r + 1, n + 4r + m] \cap \mathbb{Z}. \tag{11.28}
$$

We have that

$$
\mathcal{J}_{I \times I} \leq C\mathcal{J}_{I_1 \times I_1}\mathcal{J}_{I_2 \times I_2}. \tag{11.29}
$$

where the determinants on the right-hand side are defined as in (11.21). The above inequality expresses the fact that the matrix entries which are relevant to compute $\mathcal{J}_{I \times I}$ are close to the diagonal. Indeed, (11.29) follows from Lemma 22 and the exponential

decay obtained in Lemma 14 (see also the remark below the proof of that lemma, which shows that $\frac{\partial V_{\text{eff}}(i)}{\partial U(j)}$ has the same decay rate as $\frac{\partial V_{\text{eff}}(i)}{\partial \omega(j)}$).

The next goal is to decouple the determinants on the right-hand side of (11.29). For that purpose, let $\{\tilde{U}(j)\}_{j \in I}$ be an arbitrary list of real numbers. With the notation of Lemma 20, let $V_{\text{eff}}(U_{I_1}, \tilde{U}_{I_2}, \omega_{I^c})$. Recall that this effective potential satisfies $U(l) = \omega(l) + \frac{g}{\lambda} V_{\text{eff}}(U_{I_1}, \tilde{U}_{I_2}, \omega_{I^c})(l)$ for $l \in I_1$ and $\tilde{U}(k) = \omega(k) + \frac{g}{\lambda} V_{\text{eff}}(U_{I_1}, \tilde{U}_{I_2}, \omega_{I^c})(k)$ for $k \in I_2$. Let $\mathcal{J}_{I_1 \times I_1}(U_{I_1}, \tilde{U}_{I_2}, \omega_{I^c})$ be the Jacobian associated to $V_{\text{eff}}(U_{I_1}, \tilde{U}_{I_2}, \omega_{I^c})$, defined as in (11.20) with V_{eff} replaced by $V_{\text{eff}}(U_{I_1}, \tilde{U}_{I_2}, \omega_{I^c})$. From now on we denote the original Jacobian $\mathcal{J}_{I_1 \times I_1}$ by $\mathcal{J}_{I_1 \times I_1}(U_{I_1}, U_{I_2}, \omega_{I^c})$ to avoid ambiguity. We claim that

$$\mathcal{J}_{I_1 \times I_1}(U_{I_1}, U_{I_2}, \omega_{I^c}) \leq C^{r+1} \mathcal{J}_{I_1 \times I_1}(U_{I_1}, \tilde{U}_{I_2}, \omega_{I^c}). \tag{11.30}$$

To verify (11.30), observe that, similarly as in (11.29), due to lemmas 22 and 14 we have that $\mathcal{J}_{I_1 \times I_1}(U_{I_1}, U_{I_2}, \omega_{I^c}) \leq C \mathcal{J}_{I_{1,1} \times I_{1,1}}(U_{I_1}, U_{I_2}, \omega_{I^c}) \mathcal{J}_{I_{1,2} \times I_{1,2}}(U_{I_1}, U_{I_2}, \omega_{I^c})$. Moreover, due to Lemma 22, we know that $\mathcal{J}_{I_{1,2} \times I_{1,2}}(U_{I_1}, U_{I_2}, \omega_{I^c}) \leq C^r$, since this is a determinant of an $r \times r$ matrix with off-diagonal exponential decay. Hence,

$$\mathcal{J}_{I_1 \times I_1}(U_{I_1}, U_{I_2}, \omega_{I^c}) \leq C^{r+1} \mathcal{J}_{I_{1,1} \times I_{1,1}}(U_{I_1}, U_{I_2}, \omega_{I^c}). \tag{11.31}$$

Due to Lemma 21(b), we have that, for $l, j \in I_1$,

$$\left| \frac{\partial V_{\text{eff}}(U_{I_1}, U_{I_2}, \omega_{I^c})(l)}{\partial U(j)} - \frac{\partial V_{\text{eff}}(U_{I_1}, \tilde{U}_{I_2}, \omega_{I^c})(l)}{\partial U(j)} \right| \leq C e^{-\nu(r+|l-j|)}. \tag{11.32}$$

Due to Lemma 22 and (11.32), we have, for some constant $C > 0$

$$\mathcal{J}_{I_{1,1} \times I_{1,1}}(U_{I_1}, U_{I_2}, \omega_{I^c}) \leq C^{n e^{-\nu r}} \mathcal{J}_{I_{1,1} \times I_{1,1}}(U_{I_1}, \tilde{U}_{I_2}, \omega_{I^c}). \tag{11.33}$$

Since, by definition, $r \geq \lfloor \delta \log n \rfloor$, we may conclude from (11.31) and (11.33) that for a suitable choice of δ (for instance, $\delta \geq \frac{1}{\nu}$ suffices) we have

$$\mathcal{J}_{I_1 \times I_1}(U_{I_1}, U_{I_2}, \omega_{I^c}) \leq C^{r+1} \mathcal{J}_{I_{1,1} \times I_{1,1}}(U_{I_1}, \tilde{U}_{I_2}, \omega_{I^c}). \tag{11.34}$$

Similar considerations as in (11.31) give

$$C^{r+1} \mathcal{J}_{I_{1,1} \times I_{1,1}}(U_{I_1}, \tilde{U}_{I_2}, \omega_{I^c}) \leq \mathcal{J}_{I_1 \times I_1}(U_{I_1}, \tilde{U}_{I_2}, \omega_{I^c}). \tag{11.35}$$

The bound (11.30) follows from (11.34) and (11.35).

In a similar fashion,

$$\mathcal{J}_{I_2 \times I_2}(U_{I_1}, U_{I_2}, \omega_{I^c}) \leq C^{r+1} \mathcal{J}_{I_2 \times I_2}(\tilde{U}_{I_1}, U_{I_2}, \omega_{I^c}) \tag{11.36}$$

where we have modified U in I_1 and replaced $V_{\text{eff}}(U_{I_1}, U_{I_2}, \omega_{I^c})$ by $V_{\text{eff}}(\tilde{U}_{I_1}, U_{I_2}, \omega_{I^c})$. Since the list $\tilde{U}_{I_1} = \{\tilde{U}(l)\}_{l \in I_1}$ was arbitrary, we may choose it to be independent of the values $U_{I_1} = \{U(l)\}_{l \in I_1}$. In words, we have managed to estimate the determinant on the larger scale by a product of two smaller determinants which are less correlated.

To obtain the analogous statement for the densities, we shall make use of the fluctuation Assumption (2.4), which implies that, for some $c > 0$,

$$\frac{\rho\left(U(k) - \frac{g}{\lambda}V_{\text{eff}}(U_{I_1}, U_{I_2}, \omega_{I^c})(k)\right)}{\rho\left(U(k) - \frac{g}{\lambda}V_{\text{eff}}(U_{I_1}, \tilde{U}_{I_2}, \omega_{I^c})(k)\right)} \leq e^{c|V_{\text{eff}}(U_{I_1}, U_{I_2}, \omega_{I^c})(k) - V_{\text{eff}}(U_{I_1}, \tilde{U}_{I_2}, \omega_{I^c})(k)||U(k)|}.$$

(11.37)

It follows from Lemma 20(b) that for $k \in I_1$

$$|V_{\text{eff}}(U_{I_1}, U_{I_2}, \omega_{I^c})(k) - V_{\text{eff},1}(U_{I_1}, \tilde{U}_{I_2}, \omega_{I^c})(k)||U(k)| \leq Ce^{-\nu|k-(n+r+1)|}.$$

(11.38)

Thus, from (11.37), we reach

$$\prod_{k \in I_1} \frac{\rho\left(U(k) - \frac{g}{\lambda}V_{\text{eff}}(U_{I_1}, U_{I_2}, \omega_{I^c})(k)\right)}{\rho\left(U(k) - \frac{g}{\lambda}V_{\text{eff}}(U_{I_1}, \tilde{U}_{I_2}, \omega_{I^c})(k)\right)} \leq C.$$

(11.39)

Similarly,

$$\prod_{k \in I_2} \frac{\rho\left(U(k) - \frac{g}{\lambda}V_{\text{eff}}(U_{I_1}, U_{I_2}, \omega_{I^c})(k)\right)}{\rho\left(U(k) - \frac{g}{\lambda}V_{\text{eff}}(\tilde{U}_{I_1}, U_{I_2}, \omega_{I^c})(k)\right)} \leq C.$$

(11.40)

The inequality (11.24) follows from the bounds on determinants (11.29), (11.30) and (11.36) combined with the density bounds (11.39) and (11.40). Since $\hat{G}(0, n; z)$ is a function of $U(0), \ldots, U(n + r)$ and $\hat{G}(n + r + 1, n + r + m; z)$ is a function of $U(n + r + 1), \ldots, U(n + r + m)$ we obtain the decoupling estimate (11.23) from (11.24). The inequalities (11.26) and (11.25) can be proven similarly, by comparing $V_{\text{eff}}(\tilde{U}_I, \omega_{I^c})$ to $V_{\text{eff}}(\tilde{U}_{I_1}, \tilde{\omega}_{I^c})$ instead and making use of Lemmas 20(a) and 21(a).

12. Hölder Continuity for the Integrated Density of States at Weak Interaction

In this section we shall address the problem of Hölder continuity for the integrated density of states of the Hubbard model with respect to energy, disorder and interaction. Our results follow from modifications of the methods in [18] and references therein after we have established the existence of a suitable conditional density as in Lemma 5.

Let's now prove Theorem 3, starting from Hölder continuity with respect to energy, Eq. (2.13). We proceed as in [18, Section 2]. For simplicity, we replace H_{Hub} by H defined in (4.1). The arguments given below will apply directly to $H_\uparrow$ and $H_\downarrow$ and, therefore, suffice to show the same result for H_{Hub}.

Since the interval K is compact, it suffices to show that $N_{\lambda,g}$ is locally Hölder continuous in K. For that purpose, let $E \in K$ be an arbitrary point in the interior of K and fix an energy interval I of length $\varepsilon > 0$ centered at $E \in \mathbb{R}$. The idea is to use the Hölder continuity of N_0 and the resolvent identity to reach the following inequality for $\varepsilon << 1$ and $|I| = \varepsilon$, where we denote by $P_\Lambda(I)$ the spectral projection of H^Λ on the interval I.

$$(1 - o(\varepsilon))\mathbb{E}\left(\text{Tr}\, P_\Lambda(I)\right) \leq C(I, \rho)\varepsilon^\alpha |\Lambda|.$$

(12.1)

Dividing both sides of (12.1) by $|\Lambda|$ and letting $|\Lambda| \to \infty$ gives (2.13). To obtain (12.1) we fix an interval J containing I with $|J|$ to be determined. We then write, with $P_{0,\Lambda}(J) = P\left(H_0^\Lambda\right)(J)$,

$$\text{Tr}(P_\Lambda(I)) = \text{Tr}(P_\Lambda(I)P_{0,\Lambda}(J)) + \text{Tr}(P_\Lambda(I)P_{0,\Lambda}(J^c)).$$

(12.2)

Note

$$\text{Tr}(P_\Lambda(I) P_{0,\Lambda}(J)) \leq \text{Tr}(P_{0,\Lambda}(J)). \tag{12.3}$$

The above inequality combined with to the Hölder continuity of N_0 with respect to $E \in \mathbb{R}$

$$|N_0(E) - N_0(E')| \leq C(I, d)|E - E'|^{\alpha_0}. \tag{12.4}$$

yields, for $|\Lambda|$ sufficiently large depending only on J,

$$\text{Tr}(P_\Lambda(I) P_{0,\Lambda}(J)) \leq C(J, d)|J|^{\alpha_0}|\Lambda|. \tag{12.5}$$

We now estimate the second term on the left-hand side of Eq. (12.2). By the resolvent identity,

$$\text{Tr}\left(P_\Lambda(I) P_{0,\Lambda}(J^c)\right) = \text{Tr}\left(P_\Lambda(I)(H - E) P_{0,\Lambda}(J^c)(H_{0,\Lambda} - E)^{-1}\right)$$
$$-\lambda \text{Tr}\left(P_\Lambda(I) U^\Lambda P_{0,\Lambda}(J^c)(H_{0,\Lambda} - E)^{-1}\right). \tag{12.6}$$

Where we have written $U = V_\omega + \frac{g}{\lambda} V_{\text{eff}}$. Moreover, using using functional calculus and that E is the center of I, we estimate the first term on the left-hand side of Eq. (12.6) by

$$\text{Tr}\left((P_\Lambda(I))(H^\Lambda - E) P_{0,\Lambda}(J^c)(H_{0,\Lambda} - E)^{-1}\right) \leq \frac{|I|}{|J| - |I|} \text{Tr}(P_\Lambda(I)). \tag{12.7}$$

Now, the second term in in Eq. (12.6) can be controlled by means of

$$-\lambda \text{Tr}\left(P_\Lambda(I) U^\Lambda P_{0,\Lambda}(J^c)(H_{0,\Lambda} - E)^{-1}\right)$$
$$= -\lambda \text{Tr}\left((H^\Lambda - E)(P_\Lambda(I)) U^\Lambda P_{0,\Lambda}(J^c)(H_{0,\Lambda} - E)^{-2}\right)$$
$$+ \lambda^2 \text{Tr}\left(U^\Lambda (P_\Lambda(I)) U^\Lambda P_{0,\Lambda}(J^c)(H_{0,\Lambda} - E)^{-2}\right).$$
$$= A + B$$

Now, because U^Λ is unbounded, we continue a slight modification of the argument in [18]. The only difference is that we bound term (A) above (which corresponds to [18, (iii) in equation (2.6)]) as

$$|\text{Tr}\left((H^\Lambda - E)(P_\Lambda(I)) U^\Lambda P_{0,\Lambda}(J^c)(H_{0,\Lambda} - E)^{-2}\right)| \leq \frac{|I|}{(|J| - |I|)^2} |\text{Tr}\left(P_\Lambda(I) U^\Lambda\right)|. \tag{12.8}$$

At this point, with an estimate analogous to the one in the proof of Proposition 3.2 in [9] we reach

$$\mathbb{E}\left(|\text{Tr} P_\Lambda(I) V_\omega|\right) \leq \lambda^{-1} \sup_{m \in \mathbb{N}} \left\{ \int_{m\varepsilon}^{(m+1)\varepsilon} \omega_j \rho(\omega_j)\, d\omega_j \right\} |\Lambda| \quad, \varepsilon = |I|. \tag{12.9}$$

Thus, with $M_1(\varepsilon) := \sup_{m\in\mathbb{N}} \left\{ \int_{m\varepsilon}^{(m+1)\varepsilon} \omega_j \rho(\omega_j)\, d\omega_j \right\}$,

$$\lambda |\mathrm{Tr}(H_\Lambda - E) P_\Lambda(I) U^\Lambda P_{0,\Lambda}(J^c)(H_0^\Lambda - E)^{-2}| \leq \frac{\lambda|I|}{(|J|-|I|)^2}\left(\frac{M_1(\varepsilon)}{\lambda} + \frac{g\|F\|_\infty}{\lambda}\right)|\Lambda|.$$
$$(12.10)$$

Similarly, with $M_2(\varepsilon) := \sup_{m\in\mathbb{N}} \left\{ \int_{m\varepsilon}^{(m+1)\varepsilon} \omega_j^2 \rho(\omega_j)\, d\omega_j \right\}$, we estimate term (B) through

$$\lambda^2 |\mathrm{Tr} U^\Lambda (P_\Lambda(I)) U^\Lambda P_{0,\Lambda}(J^c)(H_{0,\Lambda} - E)^{-2}|$$
$$\leq \frac{4\lambda^2}{(|J|-|I|)^2}\left(\frac{M_2(\varepsilon)}{\lambda}|\Lambda| + \frac{g^2}{\lambda^2}\mathrm{Tr}(P_\Lambda(I))\right). \qquad (12.11)$$

Due Lemma 5 and the Wegner estimate (see [5, theorem 4.1]) we conclude that

$$\mathrm{Tr}(P_\Lambda(I)) \leq \frac{C}{\lambda}|I||\Lambda|. \qquad (12.12)$$

Choosing the interval J such that $|J| = \varepsilon^\delta$ for $\delta < 1$, keeping in mind the assumption $g^2 < \lambda$, combining the bounds (12.5), (12.7), (12.10), (12.11), (12.12) and optimizing over δ gives $\delta = \frac{1}{2+\alpha_0}$. Therefore, we reach (12.1) for $\alpha \in [0, \frac{\alpha_0}{2+\alpha_0}]$ and (2.13) is proven.

To show (IDS_2) we follow the proof of theorem 1.2 in [18]. We fix $\lambda, \lambda' \in J$ and $E \in \mathrm{Int}\, K$. As explained in [18], using Hölder continuity with respect to energy given by Eq. (2.13), trace identities and ergodicity of $H_{\lambda,g}$ and $H_{\lambda',g'}$, it suffices to estimate $\mathbb{E}\left(\mathrm{Tr}\, P_0 \varphi(H_{\lambda,g})(\varphi(H_{\lambda,g}) - \varphi(H_{\lambda',g'}))P_0\right)$ where φ is a smooth function such that

$$\begin{cases} \varphi \equiv 1 \text{ on } (-\infty, E], \\ \varphi \equiv 0 \text{ on } (-\infty, E + |\lambda - \lambda'|^\delta + |g - g'|^\delta)^c, \\ \|\varphi^{(j)}\|_\infty \leq C\left(|\lambda - \lambda'|^\delta + |g - g'|^\delta\right)^{-j}, \ j = 1, 2 \ldots, 3d+4 \end{cases} \qquad (12.13)$$

with $\delta > 0$ to be determined. The need for a high regularity of φ is due to the fact that the random potential V_ω may be unbounded. Let $\tilde{\varphi}$ be an almost analytic extension of φ of order $3 + 3d$. In particular, $\tilde{\varphi}$ is defined in a complex neighborhood of the support of φ and if $z = E + i\eta$ we have that

$$|\partial_{\bar{z}}\tilde{\varphi}(z)| \leq |\eta|^{3d+3}|\varphi^{(3d+4)}(E)|. \qquad (12.14)$$

By the Helffer-Sjöstrand formula,

$$\mathrm{Tr}\left(P_0\varphi(H_{\lambda,g})(\varphi(H_{\lambda,g}) - \varphi(H_{\lambda',g'}))P_0\right)$$
$$= \frac{1}{\pi}\int_{\mathbb{C}} \partial_{\bar{z}}\tilde{\varphi}\, \mathrm{Tr}\, P_0\varphi(H_{\lambda,g})R_{\lambda,g}(z)\left(\lambda' U_{\lambda',g'} - \lambda U_{\lambda,g}\right) R_{\lambda',g'}(z) P_0\, d^2 z$$
$$= \frac{(\lambda' - \lambda)}{\pi}\int_{\mathbb{C}} \partial_{\bar{z}}\tilde{\varphi}\, \mathrm{Tr}\, P_0\varphi(H_{\lambda,g})R_{\lambda,g}(z) V_\omega R_{\lambda',g'}(z) P_0\, d^2 z$$
$$+ \frac{(g' - g)}{\pi}\int_{\mathbb{C}} \partial_{\bar{z}}\tilde{\varphi}\, \mathrm{Tr}\, P_0\varphi(H_{\lambda,g})R_{\lambda,g}(z) V_{\mathrm{eff},\lambda}(g) R_{\lambda',g'}(z) P_0\, d^2 z$$
$$+ \frac{g'}{\pi}\int_{\mathbb{C}} \partial_{\bar{z}}\tilde{\varphi}\, \mathrm{Tr}\, P_0\varphi(H_{\lambda,g})R_{\lambda,g}(z) \left(V_{\mathrm{eff},\lambda'}(g') - V_{\mathrm{eff},\lambda}(g)\right) R_{\lambda',g'}(z) P_0\, d^2 z.$$

Since the last two terms enjoy a better modulus of Hölder continuity (since they do not involve V_ω) and can be treated as in [18], we shall only estimate the first of the above integrals. By the resolvent identity,

$$R_{\lambda,g}(z)V_\omega R_{\lambda',g'}(z) = R_{\lambda,g}(z)V_\omega R_{\lambda,g}(z) + (\lambda' - \lambda)R_{\lambda,g}(z)V_\omega R_{\lambda',g'}(z)V_\omega R_{\lambda,g}(z)$$
$$+ (g' - g)R_{\lambda,g}(z)V_\omega R_{\lambda',g'}(z)V_{\text{eff},\lambda'}(g')R_{\lambda,g}(z)$$
$$+ g R_{\lambda,g}(z)V_\omega R_{\lambda',g'}(z)(V_{\text{eff},\lambda'}(g') - V_{\text{eff},\lambda}(g))R_{\lambda,g}(z).$$

The above considerations lead to a perturbative expansion of $\frac{(\lambda'-\lambda)}{\pi}\int_{\mathbb{C}} \partial_{\overline{z}}\tilde{\varphi} \operatorname{Tr} P_0 \varphi(H_{\lambda,g})R_{\lambda,g}(z)V_\omega R_{\lambda',g'}(z)P_0\, d^2z$ into four terms. We will show below that each of them can be bounded in terms of powers of either $|\lambda - \lambda'|$ or $|g - g'|$. We start by estimating $\mathbb{E}\left(\left|\frac{(\lambda-\lambda')}{\pi}\int_{\mathbb{C}} \partial_{\overline{z}}\tilde{\varphi} \operatorname{Tr} P_0 \varphi(H_{\lambda,g})R_{\lambda,g}(z)V_\omega R_{\lambda',g'}(z)V_\omega R_{\lambda,g}(z) P_0\, d^2z\right|\right)$ with a slight modification of equation (3.15) in [18] since V_ω is unbounded. By the Combes–Thomas bound, Eq. (12.14) and the choice of φ

$$\mathbb{E}\left(\left|\frac{(\lambda-\lambda')^2}{\pi}\int_{\mathbb{C}} \partial_{\overline{z}}\tilde{\varphi} \operatorname{Tr} P_0 \varphi(H_{\lambda,g})R_{\lambda,g}(z)V_\omega R_{\lambda',g'}(z)V_\omega R_{\lambda,g}(z) P_0\, d^2z\right|\right)$$
$$\leq C(d)\left(1 + \mathbb{E}^2(|V_\omega|)\right)\frac{|\lambda - \lambda'|^2}{(|\lambda - \lambda'|^\delta + |g - g'|^\delta)^{3d+4}}.$$

Similarly,

$$\mathbb{E}\left(\left|\frac{(\lambda-\lambda')(g-g')}{\pi}\right.\right.$$
$$\left.\left.\int_{\mathbb{C}} \partial_{\overline{z}}\tilde{\varphi} \operatorname{Tr} P_0 \varphi(H_{\lambda,g})R_{\lambda,g}(z)V_{\text{eff},\lambda'}(g')R_{\lambda',g'}(z)V_\omega R_{\lambda,g}(z) P_0\, d^2z\right|\right)$$
$$\leq C(d)\left(1 + \mathbb{E}(|V_\omega|)\right)\frac{|\lambda - \lambda'||g - g'|}{(|\lambda - \lambda'|^\delta + |g - g'|^\delta)^{3d+4}}.$$

Moreover, using Lemma 17 with the the explicit dependence on ω given there, we obtain

$$\mathbb{E}\left(\left|\frac{g(\lambda-\lambda')}{\pi}\int_{\mathbb{C}} \partial_{\overline{z}}\tilde{\varphi} \operatorname{Tr} P_0 \varphi(H_{\lambda,g})R_{\lambda,g}(z)V_\omega R_{\lambda',g'}(z)(V_{\text{eff},\lambda'}(g')\right.\right.$$
$$\left.\left.-V_{\text{eff},\lambda}(g))R_{\lambda,g}(z)P_0\, d^2z\right|\right)$$
$$\leq C(d)\left(1 + \mathbb{E}(|V_\omega|)\right)\frac{|g||\lambda - \lambda'|(|g - g'| + |\lambda - \lambda'|)}{(|\lambda - \lambda'|^\delta + |g - g'|^\delta)^{3+3d}}.$$

Using the same arguments as in [18, Equations 3.17 and 3.18] we see that

$$\left|\frac{(\lambda'-\lambda)}{\pi}\int_{\mathbb{C}} \partial_{\overline{z}}\tilde{\varphi} \operatorname{Tr} P_0 \varphi(H_{\lambda,g})R_{\lambda,g}(z)V_\omega R_{\lambda',g'}(z)P_0\, d^2z\right|$$

can be bounded from above by

$$|\lambda - \lambda'| |\mathbb{E}\left(\mathrm{Tr}(P_0 \varphi(H_{\lambda,g}) R_{\lambda,g}(z) V_\omega R_{\lambda,g}(z) P_0)\right)| \leq \frac{C|\lambda - \lambda'| \mathbb{E}\left(|V_\omega|\right)}{(|\lambda - \lambda'|^\delta + |g - g'|^\delta)}.$$

$$(12.15)$$

Finally, we conclude that $|N_{\lambda,g}(E) - N_{\lambda',g'}(E)|$ is bounded from above by

$$C(\alpha_0, d, J, K)\left(|\lambda - \lambda'|^{\delta\alpha} + |g - g'|^{\delta\alpha} + |\lambda - \lambda'|^{2-(3d+4)\delta}\right.$$
$$\left. + |g - g'|^{2-(3d+4)\delta} + |\lambda - \lambda'|^{1-\delta} + |g - g'|^{1-\delta}\right).$$

Choosing $\delta = \frac{2}{\alpha+3d+4}$ we obtain, for any $\beta \in [0, \frac{2}{\alpha+3d+4}]$,

$$|N_{\lambda,g}(E) - N_{\lambda',g'}(E)| \leq C(\alpha_0, d, J, K)\left(|\lambda - \lambda'|^\beta + |g - g'|^\beta\right)$$

finishing the proof of Theorem 3.

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