HOLOMORPHIC OPEN BOOK DECOMPOSITIONS

CASIM ABBAS

Abstract

Giroux showed that every contact structure on a closed 3-dimensional manifold is supported by an open book decomposition. We extend this result by showing that the open book decomposition can be chosen in such a way that the pages are solutions to a homological perturbed holomorphic curve equation.

Contents

1. Introduction

This paper is the starting point of a larger program by the author, Hofer, and Lisi investigating a perturbed holomorphic curve equation in the symplectization of a 3-dimensional contact manifold (see [6], [7]). One aim of this program is to provide an alternative proof of the Weinstein conjecture in dimension 3 as outlined in [4] complementing Taubes's gauge theoretical proof (see [33], [34]). A special case of this paper's main result has been used in the proof of the Weinstein conjecture for planar contact structures in [4]. Another reason for studying this equation is to construct foliations by surfaces of section with nontrivial genus. This is usually impossible to do with the unperturbed holomorphic curve equation since solutions generically do not exist.

Consider a closed 3-dimensional manifold M equipped with a contact form λ. This is a 1-form which satisfies $\lambda \wedge d\lambda \neq 0$ at every point of M. We denote the associated contact structure by $\xi = \ker \lambda$, and we denote the Reeb vector field by X_λ.

DUKE MATHEMATICAL JOURNAL
Vol. 158, No. 1, © 2011 DOI 10.1215/00127094-1276301
Received 20 July 2009. Revision received 8 July 2010.
2010 *Mathematics Subject Classification.* Primary 53D35; Secondary 35J62.

Recall that the Reeb vector field is defined by the two equations

$$i_{X_\lambda} d\lambda = 0 \qquad \text{and} \qquad i_{X_\lambda}\lambda = 1.$$

Definition 1.1 (Open book decomposition)
Assume that $K \subset M$ is a link in M and that $\tau : M\backslash K \to S^1$ is a fibration so that the fibers $F_\vartheta = \tau^{-1}(\vartheta)$ are interiors of compact embedded surfaces $\bar{F}_\vartheta$ with $\partial\bar{F}_\vartheta = K$, where ϑ is the coordinate along K. We also assume that K has a tubular neighborhood $K \times D$, $D \subset \mathbb{R}^2$ being the open unit disk, such that τ restricted to $K \times (D\backslash\{0\})$ is given by $\tau(\vartheta, r, \phi) = \phi$, where (r, ϕ) are polar coordinates on D. Then we call τ an *open book decomposition* of M, the link K is called the *binding* of the open book decomposition, and the surfaces F_ϑ are called the *pages* of the open book decomposition.

It is a well-known result in 3-dimensional topology that every closed 3-dimensional orientable manifold admits an open book decomposition. Indeed, Alexander proved the following theorem in 1923 (see [10], [30]):

THEOREM 1.2
Every closed, orientable manifold M of dimension 3 is diffeomorphic to

$$W(h) \cup_{\mathrm{Id}} (\partial W \times D^2),$$

where D^2 is the closed unit disk in $\mathbb{R}^2$, where W is an orientable surface with boundary, and where $h : W \to W$ is an orientation-preserving diffeomorphism which restricts to the identity near ∂W. Here $W(h)$ denotes the manifold obtained from $W \times [0, 2\pi]$ by identifying $(x, 0)$ with $(h(x), 2\pi)$.

The above decomposition is an open book decomposition, and the pages are given by

$$F_\vartheta := (W \times \{\vartheta\}) \cup_{\mathrm{Id}} (\partial W \times I_\vartheta), \quad 0 \le \vartheta < 2\pi,$$

where $I_\vartheta := \{re^{i\vartheta} \in D^2 \mid 0 < r < 1\}$, and the binding is given by $K = \partial W \times \{0\}$. Note that we allow ∂W to be disconnected.

Giroux introduced the notion of an open book decomposition supporting a contact structure.

Definition 1.3 (Supporting open book decomposition; see [16])
Assume that M is a closed 3-dimensional manifold endowed with a contact form λ. Let τ be an open book decomposition with binding K. We say that τ *supports the contact structure* ξ if there exists a contact form λ' with the same kernel as λ so that

$d\lambda'$ induces an area form on each fiber F_ϑ with K consisting of closed orbits of the Reeb vector field $X_{\lambda'}$, and λ' orients K as the boundary of $(F_\vartheta, d\lambda')$.

We refer to a contact form λ' above as a *Giroux contact form*. Note that λ' is not unique and that it is in general different from the original contact form λ. The following theorem by Giroux guarantees existence of such open book decompositions, and it contains a uniqueness statement as well (see also [16, Proposition 2]).

THEOREM 1.4 ([16, Theorem 3])
Every co-oriented contact structure $\xi = \ker\lambda$ *on a closed* 3*-dimensional manifold is supported by some open book. Conversely, if two contact structures are supported by the same open book, then they are diffeomorphic.*

In the topological category, it is possible to modify an open book decomposition such that the pages of the new decomposition have lower genus at the expense of increasing the number of connected components of K. It was not known for some time whether a similar statement could also be made in the context of supporting open book decompositions. In particular, it was unclear whether every contact structure was supported by an open book decomposition whose pages were punctured spheres (planar pages). The author and his collaborators could resolve the Weinstein conjecture for contact forms inducing a *planar contact structure* in 2005 (see [4]). So the question of whether all contact structures are planar became a priority, which prompted Etnyre to address it in [14]. He showed that overtwisted contact structures always admit supporting open book decompositions with planar pages, but many contact structures do not. Since then, planar open book decompositions have become an important tool in contact geometry.

In this paper, we will prove that every contact structure has a supporting open book decomposition such that the pages solve a *homological perturbed Cauchy-Riemann type equation* which we now describe after introducing some notation. We write $\pi_\lambda = \pi : TM \to \xi$ for the projection along the Reeb vector field X_λ. Fix a complex multiplication $J : \xi \to \xi$ so that the map $\xi \oplus \xi \to \mathbb{R}$, defined by

$$(h, k) \to d\lambda(h, Jk),$$

defines a positive definite metric on the fibers. We call such complex multiplications *compatible* (with $d\lambda$). The equation of interest here is the following nonlinear first-order elliptic system. The solutions consist of 5-tuplets $(S, j, \Gamma, \tilde{u}, \gamma)$ where (S, j) is a closed Riemann surface with complex structure j, $\Gamma \subset S$ is a finite subset, $\tilde{u} = (a, u) : \dot{S} \to \mathbb{R} \times M$ is a proper map with $\dot{S} = S \setminus \Gamma$, and γ is a 1-form on S so that

$$\begin{cases} \pi \circ Tu \circ j = J \circ \pi \circ Tu \ \text{ on } \ \dot{S} \\ (u^*\lambda) \circ j = da + \gamma \ \text{ on } \ \dot{S} \\ d\gamma = d(\gamma \circ j) = 0 \ \text{ on } \ S \\ E(\tilde{u}) < \infty. \end{cases} \tag{1.1}$$

Here the *energy* $E(\tilde{u})$ is defined by

$$E(\tilde{u}) = \sup_{\varphi \in \Sigma} \int_{\dot{S}} \tilde{u}^* d(\varphi\lambda),$$

where Σ consists of all smooth maps $\varphi : \mathbb{R} \to [0, 1]$ with $\varphi'(s) \geq 0$ for all $s \in \mathbb{R}$.

Note that equation (1.1) reduces to the usual pseudoholomorphic curve equation in the symplectization $\mathbb{R} \times M$ if we set $\gamma = 0$. The following proposition, which is a modification of a result by Hofer [17], shows that solutions to problem (1.1) approach cylinders over periodic orbits of the Reeb vector field.

PROPOSITION 1.5
Let (M, λ) be a closed 3-dimensional manifold equipped with a contact form λ. Then the associated Reeb vector field has periodic orbits if and only if the associated PDE *problem (1.1) has a nonconstant solution.*

Proof
Let $(S, j, \Gamma, \tilde{u}, \gamma)$ be a nonconstant solution of (1.1). If $\Gamma \neq \emptyset$, then the results in [17] imply that, near a puncture, the solution is asymptotic to a periodic orbit (see also [3] for a complete proof). Here we use the fact that γ is exact near the punctures. The aim now is to show that, in the absence of punctures, the map a is constant while the image of u lies on a periodic Reeb orbit. Assume that $\Gamma = \emptyset$. Since

$$u^*\lambda = -da \circ j - \gamma \circ j,$$

we find, after applying d, that

$$\Delta_j a = -d(da \circ j) = u^* d\lambda.$$

In view of the equation $\pi \circ Tu \circ j = J \circ \pi \circ Tu$, we see that $u^* d\lambda$ is a nonnegative integrand. Applying Stokes's theorem, we obtain $\int_S u^* d\lambda = 0$, implying that

$$\pi \circ Tu \equiv 0.$$

Hence a is a harmonic function on S and therefore constant. So far, we also know that the image of u lies on a Reeb trajectory, and it remains to show that this trajectory is actually periodic.

Let $\tau : \tilde{S} \to S$ be the universal covering map. The complex structure j lifts to a complex structure $\tilde{j}$ on $\tilde{S}$. Now pick smooth functions f, g on $\tilde{S}$ such that

$$dg = \tau^*\gamma =: \tilde{\gamma}, \qquad -df = \tau^*(\gamma \circ j) = \tilde{\gamma} \circ \tilde{j}.$$

Then the map $u \circ \tau : \tilde{S} \to M$ satisfies

$$(u \circ \tau)^*\lambda \;=\; df.$$

The image of $u \circ \tau$ lies on a trajectory x of the Reeb vector field in view of

$$D(u \circ \tau)(z)\zeta \;=\; Df(z)\zeta \cdot X_\lambda\big((u \circ \tau)(z)\big),$$

hence $(u \circ \tau)(z) = x(h(z))$ for some smooth function h on $\tilde{S}$, and it follows that, after maybe adding a constant to f, we have

$$(u \circ \tau)(z) \;=\; x\big(f(z)\big).$$

The function f does not descend to S. If it did, it would have to be constant since it is harmonic. On the other hand, this would imply that u is constant in contradiction to our assumption that it is not. Therefore, there is a point $q \in S$ and two lifts $z_0, z_1 \in \tilde{S}$ such that $f(z_0) > f(z_1)$. Let $\ell : S^1 \to S$ be a loop which lifts to a path $\alpha : [0, 1] \to \tilde{S}$ with $\alpha(0) = z_0$ and $\alpha(1) = z_1$. Considering the map

$$v := u \circ \ell : S^1 \longrightarrow M,$$

we see that $v(t) = (u \circ \tau \circ \alpha)(t) = x\big(f(\alpha(t))\big)$ and $x(f(z_0)) = x(f(z_1))$, that is, the trajectory x is a periodic orbit. Hence the image of u is a periodic orbit for the Reeb vector field. □

The following is the main result of this paper.

THEOREM 1.6

Let M be a closed 3-dimensional manifold, and let λ' be a contact form on M. Then the following holds for a suitable contact form $\lambda = f\lambda'$, where f is a positive function on M. There exists a smooth family $(S, j_\tau, \Gamma_\tau, \tilde{u}_\tau = (a_\tau, u_\tau), \gamma_\tau)_{\tau \in S^1}$ of solutions to (1.1) for a suitable compatible complex structure $J : \ker\lambda \to \ker\lambda$ such that

- *all maps u_τ have the same asymptotic limit K at the punctures, where K is a finite union of periodic trajectories of the Reeb vector field X_λ;*

- $u_\tau(\dot S) \cap u_{\tau'}(\dot S) = \emptyset$ *if* $\tau \neq \tau'$;
- $M \backslash K = \bigcup_{\tau \in S^1} u_\tau(\dot S)$;
- *the projection* P *onto* S^1 *defined by* $p \in u_\tau(\dot S) \mapsto \tau$ *is a fibration;*
- *the open book decomposition given by* (P, K) *supports the contact structure* $\ker \lambda$*, and* λ *is a Giroux form.*

Here is a very brief outline of the argument. The reader is invited to skip forward to Section 4 to see in more detail how all the partial results of this paper are tied together to prove the main result. In Section 2, we find a Giroux contact form which has a certain normal form near the binding. Following an argument by Wendl ([39], [38]), we will then *almost* be able to turn the Giroux leaves into solutions of (1.1) without harmonic form except for the fact that we have to accept a confoliation form instead of a contact form. Pick one of these Giroux leaves as a starting point. The next step is to prove a result which permits us to perturb the Giroux leaf into a genuine solution of (1.1) while simultaneously perturbing the confoliation form slightly into a contact form. This is where the harmonic form in (1.1) is required. We actually obtain a local family of nearby solutions, not just one. In Section 3, we prove a compactness result which extends the local family of solutions into a global one. The remarkable fact is that there is a compactness result in the context of this paper, although there is none in general for the perturbed holomorphic curve equation. The special circumstances in this paper imply a crucial a priori bound which implies that a sequence of solutions has a pointwise convergent subsequence with a measurable limit. The objective is then to show that the regularity of this limit is much better, that it is actually smooth.

We consider two solutions $(S, j, \Gamma, \tilde u, \gamma)$ and $(S', j', \Gamma', \tilde u', \gamma')$ equivalent if there exists a biholomorphic map $\phi : (S, j) \to (S', j')$ mapping Γ to Γ' (preserving the enumeration) so that $\tilde u' \circ \phi = \tilde u$. We will often identify a solution $(S, j, \Gamma, \tilde u, \gamma)$ of (1.1) with its equivalence class $[S, j, \Gamma, \tilde u, \gamma]$. We note that we have a natural $\mathbb{R}$-action on the solution set by associating to $c \in \mathbb{R}$ and $[S, j, \Gamma, \tilde u, \gamma]$ the new solution

$$c + [S, j, \Gamma, \tilde u, \gamma] = [S, j, \Gamma, (a + c, u), \gamma], \quad \tilde u = (a, u).$$

A crucial concept for our discussion is the notion of a finite energy foliation $\mathcal{F}$.

Definition 1.7 (Finite energy foliation)
A foliation $\mathcal{F}$ of $\mathbb{R} \times M$ is called a *finite energy foliation* if every leaf F is the image of an embedded solution $[S, j, \Gamma, \tilde u, \gamma]$ of the equations (1.1), that is,

$$F = \tilde u(\dot S),$$

so that $u(\dot{S}) \subset M$ is transverse to the Reeb vector field, and for every leaf $F \in \mathcal{F}$ we also have $c + F \in \mathcal{F}$ for every $c \in \mathbb{R}$; that is, the foliation is $\mathbb{R}$-invariant.

We recall the concept of a global surface of section. Let M be a closed 3-manifold, and let X be a nowhere-vanishing smooth vector field.

Definition 1.8 (Surface of section)

(a) A local surface of section for (M, X) consists of an embedded compact surface $\Theta \subset M$ with boundary, so that $\partial\Theta$ consists of a finite union of periodic orbits (called the *binding orbits*). In addition, the interior $\dot{\Theta} = \Theta \setminus \partial\Theta$ is transverse to the flow.

(b) A local surface of section is called a *global surface of section* if, in addition, every orbit other than a binding orbit hits $\dot{\Theta}$ in forward and backward time. Furthermore, the globally defined return map $\Psi : \dot{\Theta} \to \dot{\Theta}$ has a bounded return time; that is, there exists a constant $c > 0$ so that every $x \in \dot{\Theta}$ hits $\dot{\Theta}$ again in forward time not exceeding c.

Using Proposition 2.5 below, the existence part of Giroux's theorem can be rephrased as follows.

THEOREM 1.9

Let M be a closed orientable 3-manifold, and let $\tilde{\lambda}$ be a contact form on M. Then there exists a smooth function $f : M \to (0, \infty)$ so that the contact form $\lambda = f\tilde{\lambda}$ has a Reeb vector field admitting a global surface of section.

Existence results for finite energy foliations with a given contact form λ are hard to come by since they usually have striking consequences. In [20] for example, Hofer, Wysocki, and Zehnder show that every compact strictly convex energy hypersurface S in $\mathbb{R}^4$ carries either two or infinitely many closed characteristics. The proof relies on constructing a special finite energy foliation. In special cases they were established by Hofer, Wysocki, and Zehnder [22] and by Wendl ([38], [40]). Proofs usually require a starting point, that is, a finite energy foliation for a slightly different situation as the given one. Then some kind of continuation argument is employed where all kinds of things can and do happen to the original foliation. In [22], the authors start with an explicit finite energy foliation for the round 3-dimensional sphere $S^3 \subset \mathbb{R}^4$ which is then deformed. Wendl's papers also use a rather special manifold as a starting point. The main result of this paper, Theorem 1.6, provides a starting finite energy foliation for any closed 3-dimensional contact manifold $(M, \ker\lambda)$ since it is obtained from deforming the leaves of Giroux's open book decomposition. The pages are usually not punctured spheres, and generically there are no pseudoholomorphic curves on punctured surfaces with genus which are transverse to the Reeb vector field. This makes the introduction of the harmonic form in (1.1) a necessity. The price to be paid is that compactness issues are more complicated.

Wendl [39] published a proof of Theorem 1.6 for the special case where $\ker\lambda$ is a planar contact structure, that is, where the surfaces $\dot{S}$ are punctured spheres. This result was outlined in [4]. Regardless of whether the contact structure is planar or not, there are two main steps in the proof: existence of a solution and compactness of a family of solutions. While the author established the compactness part for Theorem 1.6 long before [4] appeared, we will use the same argument described by Wendl in [39] for the existence part since it simplifies the proof considerably.

The main theorem of this article was the first step in the proof of the Weinstein conjecture for the planar case in [4]. Recall that the Weinstein conjecture [38, p. 358] states the following: *Every Reeb vector field X on a closed contact manifold M admits a periodic orbit.*

In fact, Weinstein added the additional hypothesis that the first cohomology group $H^1(M, \mathbb{R})$ with real coefficients vanishes, but there seems to be no indication that this additional hypothesis is needed.

Moreover, Theorem 1.6 is also the starting point for the construction of global surfaces of section in the forthcoming paper [7]. Another application will be an alternative proof of the Weinstein conjecture in dimension 3 (see [7]) as outlined in [4]. This complements Taubes's recent proof of the Weinstein conjecture in dimension 3 using a perturbed version of the Seiberg-Witten equations (see [33], [34]). The main issue with the homological perturbed holomorphic curve equation (1.1) is that there is no natural compactification of the space of solutions unless the harmonic forms are uniformly bounded. In the forthcoming papers [6] and [7] the lack of compactness is investigated, and bounds for the harmonic forms are derived in particular cases.

2. Existence and local foliations

2.1. Local model near the binding orbits

We use the same approach as in [39] and [38] to prove existence of a solution to (1.1). Given a closed contact 3-manifold (M, ξ), Giroux's theorem implies that there is an open book decomposition as in Theorem 1.2 supporting ξ. On the other hand, any other contact structure ξ' supported by the same open book is diffeomorphic to ξ. Starting with an open book decomposition for M, we construct a contact structure supported by Giroux contact form λ which has a certain normal form near the binding.

Definition 2.1
Let $\theta \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$ denote polar coordinates on the unit disk $D \subset \mathbb{R}^2$ by (r, ϕ), and let $\gamma_1, \gamma_2 : [0, +\infty) \to \mathbb{R}$ be smooth functions. A 1-form

$$\lambda = \gamma_1(r)\, d\theta + \gamma_2(r)\, d\phi$$

is called a *local model near the binding* if the following conditions are satisfied:

(1) the functions γ_1, γ_2, and $\gamma_2(r)/r^2$ are smooth if considered as functions on the disk D (in particular, $\gamma_1'(0) = \gamma_2'(0) = \gamma_2(0) = 0$);
(2) $\mu(r) := \gamma_1(r)\gamma_2'(r) - \gamma_1'(r)\gamma_2(r) > 0$ if $r > 0$;
(3) $\gamma_1(0) > 0$ and $\gamma_1'(r) < 0$ if $r > 0$;
(4) $\lim_{r\to 0}(\mu(r)/r) = \gamma_1(0)\gamma_2''(0) > 0$;
(5) $\kappa := (\gamma_1''(0)/\gamma_2''(0)) \notin \mathbb{Z}$ and $\kappa \leq -1/2$;
(6) $A(r) = (1/\mu^2(r))(\gamma_2''(r)\gamma_1'(r) - \gamma_1''(r)\gamma_2'(r))$ is of order r for small $r > 0$.

We explain some of the conditions above. First, since

$$\lambda \wedge d\lambda = \mu(r) d\theta \wedge dr \wedge d\phi = \frac{\mu(r)}{r} d\theta \wedge dx \wedge dy,$$

the form λ is a contact form on $S^1 \times D$. The Reeb vector field is given by

$$X(\theta, r, \phi) = \frac{\gamma_2'(r)}{\mu(r)}\frac{\partial}{\partial\theta} - \frac{\gamma_1'(r)}{\mu(r)}\frac{\partial}{\partial\phi} =: \alpha(r)\frac{\partial}{\partial\theta} + \beta(r)\frac{\partial}{\partial\phi}.$$

The trajectories of X all lie on tori $T_r = S^1 \times \partial D_r$:

$$\theta(t) = \theta_0 + \alpha(r)\, t, \qquad \phi(t) = \phi_0 + \beta(r)\, t. \tag{2.1}$$

We compute

$$\lim_{r\to 0} \alpha(r) = \lim_{r\to 0} \frac{\gamma_2''(r)}{\mu'(r)} = \frac{\gamma_2''(0)}{\gamma_1(0)\gamma_2''(0)} = \frac{1}{\gamma_1(0)}$$

and

$$\lim_{r\to 0} \beta(r) = -\lim_{r\to 0} \frac{\gamma_1''(r)}{\mu'(r)} = -\frac{\gamma_1''(0)}{\gamma_1(0)\gamma_2''(0)}.$$

Recalling that $\frac{\partial}{\partial\phi} = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$, we obtain for $r = 0$

$$X = \frac{1}{\gamma_1(0)}\frac{\partial}{\partial\theta},$$

that is, the central orbit has minimal period $2\pi\gamma_1(0)$. If the ratio $\alpha(r)/\beta(r)$ is irrational then the torus T_r carries no periodic trajectories. Otherwise, T_r is foliated with periodic trajectories of minimal period

$$\tau = \frac{2\pi m}{\alpha} = \frac{2\pi n}{\beta},$$

where $\alpha/\beta = m/n$ or $\beta/\alpha = n/m$ for suitable integers m, n (choose whatever makes sense if either α or β is zero). We calculate

$$\begin{aligned}
\lim_{r\to 0}\frac{d\alpha}{dr} &= \lim_{r\to 0}\frac{\gamma_2''(r)\mu(r) - \gamma_2'(r)\mu'(r)}{\mu^2(r)} \\
&= \lim_{r\to 0}\frac{\gamma_2'''(r)}{2\mu'(r)} - \lim_{r\to 0}\frac{\mu''(r)}{2\mu'(r)}\,\frac{\gamma_2'(r)}{r}\,\frac{r}{\mu(r)} \\
&= \frac{\gamma_2'''(0)}{2\mu'(0)} - \frac{\gamma_1(0)\gamma_2'''(0)\gamma_2''(0)}{2(\mu'(0))^2} \\
&= 0
\end{aligned}$$

since $\mu''(0) = \gamma_1(0)\gamma_2'''(0)$ and $\mu'(0) = \gamma_1(0)\gamma_2''(0) > 0$. Converting to Cartesian coordinates on the disk, we get

$$X(\theta, x, y) = \alpha(x, y)\frac{\partial}{\partial\theta} - \beta(x, y)\, y\,\frac{\partial}{\partial x} + \beta(x, y)\, x\,\frac{\partial}{\partial y},$$

and linearizing the Reeb vector field along the center orbit yields

$$DX(\theta, 0, 0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\beta(0) \\ 0 & \beta(0) & 0 \end{pmatrix}.$$

The linearization of the Reeb flow is given by

$$D\phi_t(\theta, 0, 0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta(0)t & -\sin\beta(0)t \\ 0 & \sin\beta(0)t & \cos\beta(0)t \end{pmatrix} \qquad (2.2)$$

with

$$\Phi(t) = e^{\beta(0)tJ}, \qquad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The spectrum of $\Phi(t)$ is given by

$$\sigma\big(\Phi(t)\big) = \{e^{\pm i\beta(0)t}\}.$$

The binding orbit has period $2\pi\gamma_1(0)$, since

$$\gamma_1(0)\beta(0) = -\frac{\gamma_1''(0)}{\gamma_2''(0)} \notin \mathbb{Z},$$

and it is nondegenerate and elliptic.

Example 2.2
For the contact form $T\,d\theta + (1/k)(x\,dy - y\,dx) = T\,d\theta + (r^2/k)d\phi$ the central orbit $S^1 \times \{0\}$ is degenerate, but

$$\lambda = (1-r^2)(T\,d\theta + \frac{r^2}{k}\,d\phi)$$

is a local model near the binding if

$$k, T > 0, \qquad kT \notin \mathbb{Z}, \qquad \text{and} \qquad kT \geq \frac{1}{2}.$$

In this case,

$$\mu(r) = \frac{2rT}{k}(1-r^2)^2 > 0 \qquad \text{and} \qquad \frac{\gamma_1''(0)}{\gamma_2''(0)} = -kT,$$

and we note that

$$A(r) = \frac{1}{\mu^2(r)}\big(\gamma_2''(r)\gamma_1'(r) - \gamma_1''(r)\gamma_2'(r)\big) = \frac{4kr}{T(1-r^2)^4}.$$

If

$$\frac{\alpha(r)}{\beta(r)} = -\frac{\gamma_2'(r)}{\gamma_1'(r)} = \frac{1-2r^2}{kT} = \frac{m}{n}$$

for integers n, m, then the invariant torus T_r is foliated with periodic orbits. The case $m = 0$ is only possible if $r = 1/\sqrt{2}$. If r is sufficiently small, then $|m| \geq 2$. Indeed, we would otherwise be able to find sequences $r_l \searrow 0$ and $\{n_l\} \subset \mathbb{Z}$ such that $kT/(1-2r_l^2) = n_l$, which is impossible. The binding orbit has period $2\pi T$ while the periodic orbits close to the binding orbit have much larger periods equal to

$$\tau = 2\pi T m \frac{(1-r^2)^2}{1-2r^2}.$$

Example 2.3
Consider the contact form $\lambda = T(1-r^2)d\theta + (r^2/k)d\phi$ on $S^1 \times D$. It is also a local model near the binding if $k, T > 0$, $kT \geq 1/2$, and kT is not an integer. We

even have $A(r) \equiv 0$. In contrast to Example 2.2, if $kT \notin \mathbb{Q}$, then the invariant tori T_r carry no periodic orbits. If $kT = n/m \in \mathbb{Q}$ but not in $\mathbb{Z}$, all invariant tori are foliated with periodic orbits of period $2\pi mT$ with $|m| \geq 2$ while the binding orbit has period $2\pi T$. The function $A(r)$ is identically zero. This is the contact form on the irrational ellipsoid in $\mathbb{R}^4$.

The following proposition is essentially due to Wendl. The construction in the proof was used by Thurston and Winkelnkemper [35] to show existence of contact forms on closed 3-manifolds.

PROPOSITION 2.4 ([39, Proposition 1])
Let M be a 3-dimensional manifold given by an open book decomposition

$$M = W(h) \cup_{\mathrm{Id}} (\partial W \times D^2)$$

as described in Theorem 1.2. We denote the pages by

$$F_\alpha := (W \times \{\alpha\}) \cup_{\mathrm{Id}} (\partial W \times I_\alpha), \quad 0 \leq \alpha < 2\pi,$$

where $I_\alpha := \{re^{i\alpha} \in D \mid 0 < r < 1\}$, and we denote the binding $\partial W \times \{0\}$ by K. Moreover, let λ_2 be a contact form on $\partial W \times D$ which is a local model near the binding on each connected component of $\partial W \times D$. Then there is a smooth family of 1-forms $(\lambda_\delta)_{0 \leq \delta < 1}$ on M such that the following hold.

- *The form λ_0 is a confoliation 1-form; that is, $\lambda_0 \wedge d\lambda_0 \geq 0$, and* $\ker \lambda_0$ *agrees with the tangent spaces to the pages F_ϑ away from the binding.*
- *For $\delta > 0$, the forms λ_δ are contact forms such that* $\ker \lambda_\delta$ *is supported by the above open book. In particular, the Reeb vector fields X_{λ_δ} are transverse to the pages F_α, and the binding K consists of periodic orbits of X_{λ_δ}.*
- *The forms λ_δ agree with the local model λ_2 near the binding. In particular, the binding orbits are nondegenerate and elliptic.*

Proof
We first construct contact/confoliation forms λ_1 on $W(h)$, depending smoothly on a parameter $\delta \geq 0$, that we control well near the boundary $\partial W(h) \approx \partial W \times S^1$. Then we glue these forms together with λ_2 in a smooth way to obtain a contact form on $W(h) \cup_{\mathrm{Id}} (\partial W \times D^2)$ for $\delta > 0$ or a confoliation form for $\delta = 0$. This procedure was used by Thurston and Winkelnkemper [35], where they showed that every open book is supported by some contact structure.

Starting with an open book as above, we can find a collar neighborhood C of ∂W so that $h(t, \theta) = (t, \theta)$ for all $(t, \theta) \in C$. Here we identify $(C, \partial W)$ with

$([0,\varepsilon]\times(\dot{\bigcup}_n S^1), \{0\}\times(\dot{\bigcup}_n S^1))$, where we take an n-fold disjoint union of circles $S^1\approx\mathbb{R}/2\pi\mathbb{Z}$ according to the number n of components of ∂W.

We claim that there is an area form Ω on W that satisfies

- $\int_W \Omega = 2\pi n$,
- $\Omega|_C = dt\wedge d\theta$.

Indeed, start with any area form Ω' so that $\int_W \Omega' = 2\pi n$. Then we have $\Omega'|_C = f'(t,\theta)dt\wedge d\theta$ with a positive smooth function f' (after switching signs if necessary). Now pick a new smooth positive function f which is equal to some constant c if $t\le(1/3)\varepsilon$ and which agrees with f' if $t\ge(2/3)\varepsilon$ so that the resulting area form Ω still satisfies $\int_W\Omega = 2\pi n$. Do one component of ∂W at a time. Rescaling the t-coordinate, we may assume that $c=1$.

Let α_1 be any 1-form on W which equals $(1+t)\,d\theta$ near ∂W. Then by Stokes's theorem we obtain

$$\int_W(\Omega-d\alpha_1)=2\pi n-\int_{\partial W}\alpha_1=2\pi n+\int_{\partial W}d\theta=0.$$

The 2-form $\Omega-d\alpha_1$ on W is closed and vanishes near ∂W. Then there exists a 1-form β on W with

$$d\beta=\Omega-d\alpha_1$$

and $\beta\equiv 0$ near ∂W. Now define $\alpha_2:=\alpha_1+\beta$. Then α_2 satisfies the following:

- $d\alpha_2$ is an area form on W inducing the same orientation as Ω, (2.3)
- $\alpha_2=(1+t)\,d\theta$ near ∂W. (2.4)

The set of 1-forms on W satisfying (2.3) and (2.4) is therefore nonempty and also convex. We define the following 1-form on $W\times[0,2\pi]$, where α is any 1-form on W satisfying (2.3) and (2.4):

$$\tilde{\alpha}(x,\tau):=\tau\alpha(x)+(2\pi-\tau)(h^*\alpha)(x).$$

This 1-form descends to the quotient $W(h)$, and the restriction to each fiber of the fiber bundle $W(h)\xrightarrow{\pi}S^1$ satisfies condition (2.3). Moreover, since $h\equiv\mathrm{Id}$ near ∂W, we have $\tilde{\alpha}(x,\tau)=2\pi(1+t)\,d\theta$ for all $(x,\tau)=((t,\theta),\tau)$ near $\partial W(h)=\partial W\times S^1$. Let $d\tau$ be a volume form on S^1. We claim that

$$\lambda_1:=-\delta\tilde{\alpha}+\pi^*d\tau$$

are contact forms on $W(h)$ whenever $\delta > 0$ is sufficiently small. Pick $(x, \tau) \in W(h)$, and let $\{u, v, w\}$ be a basis of $T_{(x,\tau)}W(h)$ with $\pi_* u = \pi_* v = 0$. Then

$$\begin{aligned}
&(\lambda_1 \wedge d\lambda_1)(x, \tau)(u, v, w) \\
&\quad = \delta^2(\tilde{\alpha} \wedge d\tilde{\alpha})(x, \tau)(u, v, w) - \delta\,[d\tau(\pi_* w)\, d\tilde{\alpha}(x, \tau)(u, v)] \\
&\quad \neq 0
\end{aligned}$$

for sufficiently small $\delta > 0$, and $d\lambda_1$ is a volume form on W. Now we have to continue the contact forms λ_1 beyond $\partial W(h) \approx \partial W \times S^1$ onto $\partial W \times D^2$. At this point it is convenient to change coordinates. We identify $C \times S^1$ with $\partial W \times (D^2_{1+\varepsilon} \backslash D^2_1)$, where D^2_ρ is the 2-disk of radius ρ. Using polar coordinates (r, ϕ) on $D^2_{1+\varepsilon}$ with $0 \le \phi \le 2\pi$ and $0 < r \le 1 + \varepsilon$, our old coordinates are related to the new ones by

$$\partial W \times (D^2_{1+\varepsilon} \backslash D^2_1) \ni (\theta, r, \phi) \approx (\theta, 1 + t, \tau) \in C \times S^1,$$

and λ_1 is given by

$$\lambda_1 = -\frac{\delta}{2\pi}\, r\, d\theta + d\phi$$

on $\partial W \times (D^2_{1+\varepsilon} \backslash D^2_1)$, with ε sufficiently small so that (2.4) holds. From now on we drop the factor $1/2\pi$, absorbing it into the constant δ. We have to extend this now smoothly to a contact form on $\partial W \times D^2_{1+\varepsilon}$ which agrees with λ_2 near $\{r = 0\}$. We set

$$\lambda = \gamma_1(r)\, d\theta + \gamma_2(r)\, d\phi,$$

where γ_1, γ_2 satisfy the conditions in Definition 2.1 for small r, say $r \le \varepsilon_0$, and

$$\gamma_1(r) = -\delta r, \qquad \gamma_2(r) = 1 \quad \text{for } r \ge 1 - \varepsilon_0.$$

If we write $\gamma(r) = \gamma_1(r) + i\gamma_2(r) = \rho(r)\, e^{i\alpha(r)}$, then

$$\mu(r) := \gamma_1(r)\gamma_2'(r) - \gamma_1'(r)\gamma_2(r) = \Re(i\gamma(r)\,\overline{\gamma'(r)}) = \rho^2(r)\alpha'(r),$$

which has to be positive. Also recall from Definition 2.1 that

$$\gamma_1(0) > 0 \qquad \text{and} \qquad \gamma_1'(r) < 0 \quad \text{if } r > 0,$$

hence the curves $\gamma = \gamma_\delta$ have to turn counterclockwise in the first quadrant starting at the point $(\gamma_1(0), 0)$ and later connecting with $(-\delta(1 - \varepsilon_0), 1)$. In the case where $\delta > 0$, the Reeb vector fields are given by

$$X_\delta(\theta, r, \phi) = \frac{\gamma_2'(r)}{\mu(r)} \frac{\partial}{\partial\theta} - \frac{\gamma_1'(r)}{\mu(r)} \frac{\partial}{\partial\phi},$$

and, in particular,

$$X_\delta(\theta, r, \phi) = \frac{\partial}{\partial \phi} \text{ for } r \geq 1 - \varepsilon_0, \tag{2.5}$$

which implies that the Reeb vector fields X_δ converge as $\delta \searrow 0$. In addition to λ_δ being contact forms for $\delta > 0$, we also want the given open book decomposition to support $\ker \lambda_\delta$, hence X_δ needs to be transverse to the pages of the open book decomposition which is equivalent to $\gamma_1'(r) \neq 0$. A curve $\gamma(r)$ fulfilling these conditions can clearly be constructed. □

The following result shows that we can always assume that a Giroux contact form is equal to any of the forms provided by Proposition 2.4.

PROPOSITION 2.5
Let M be a closed 3-dimensional manifold with contact structure ξ. Then, for every $\delta > 0$, there is a diffeomorphism $\varphi_\delta : M \to M$ such that $\ker \lambda_\delta = \varphi_* \xi$ where λ_δ is given by Proposition 2.4.

Proof
Existence of an open book decomposition supporting ξ follows from the existence part of Giroux's theorem. On the other hand, Proposition 2.4 yields contact forms λ_δ such that $\ker \lambda_\delta$ is also supported by the same open book decomposition as ξ for any $\delta > 0$. By the uniqueness part of Giroux's theorem, ξ and $\ker \lambda_\delta$ are diffeomorphic. □

It follows from our previous construction of the forms λ_δ that λ_0 satisfies $\lambda_0 \wedge d\lambda_0 > 0$ on $\partial W \times D_{1-\varepsilon_0}$ and that $\lambda_0 = d\phi$ otherwise. For $\delta \to 0$, the Reeb vector fields X_δ will converge to some vector field X_0, which is the Reeb vector field of λ_0 if $r < 1 - \varepsilon_0$ and which equals $\frac{\partial}{\partial \phi}$ everywhere else.

PROPOSITION 2.6
Let M be a closed 3-dimensional manifold with an open book decomposition and a family of 1-forms λ_δ, $\delta \geq 0$ as in Proposition 2.4. Then we have

- *a smooth family $(\tilde{J}_\delta)_{\delta \geq 0}$ of almost-complex structures on $T(\mathbb{R} \times M)$ which are $\mathbb{R}$-independent and which satisfy $\tilde{J}_\delta(X_\delta) = -\partial/\partial\tau$, where τ denotes the coordinate on $\mathbb{R}$, so that $J_\delta := \tilde{J}_\delta|_{\ker \lambda_\delta}$ are $d\lambda_\delta$-compatible whenever λ_δ is a contact form;*
- *a parameterization of the Giroux leaves $u_\alpha : \dot{S} \to M$, $\alpha \in [0, 2\pi]$, where $\dot{S} = S \backslash \{p_1, \ldots, p_n\}$ and where S is a closed surface; and*
- *a smooth family of smooth functions $a_\alpha : \dot{S} \to \mathbb{R}$*

such that $\tilde{u}_\alpha = (a_\alpha, u_\alpha) : \dot{S} \to \mathbb{R} \times M$ is a family of embedded $\tilde{J}_0$-holomorphic curves for a suitable smooth family of complex structures j_α on S which restrict to *the standard complex structure on the cylinder* $[0,+\infty)\times S^1$ *after introducing polar coordinates near the punctures. Moreover, all the punctures are positive** *the family* $(\tilde{u}_\alpha)_{0\le\alpha\le 2\pi}$ *is a finite energy foliation, and the curves* $\tilde{u}_\alpha$ *are* $\tilde{J}_\delta$*-holomorphic near the punctures.*

Proof

We parameterize the leaves of the open book decomposition $u_\alpha : \dot{S}\to M$, $0\le\alpha<2\pi$, and we assume that they look as follows near the binding:

$$\begin{gathered} u_\alpha : [0,+\infty)\times S^1 \longrightarrow S^1\times D_1,\\ u_\alpha(s,t) = \big(t, r(s)e^{i\alpha}\big), \end{gathered} \tag{2.6}$$

where r are smooth functions with $\lim_{s\to\infty} r(s) = 0$ to be determined shortly. We use the notation (r,ϕ) for polar coordinates on the disk $D = D_1$. We identify some neighborhood U of the punctures of $\dot{S}$ with a finite disjoint union of half-cylinders $[0,+\infty)\times S^1$. Recall that the binding orbit is given by

$$x(t) = \Big(\frac{t}{\gamma_1(0)}, 0, 0\Big), \quad 0\le t\le 2\pi\gamma_1(0)$$

and that it has minimal period $T = 2\pi\gamma_1(0)$. We define smooth functions $a_\alpha : \dot{S}\to\mathbb{R}$ by

$$a_\alpha(z) := \begin{cases} \int_0^s \gamma_1\big(r(s')\big)\,ds' & \text{if } z=(s,t)\in[0,+\infty)\times S^1\subset U,\\ 0 & \text{if } z\notin U \end{cases}$$

so that

$$u_\alpha^*\lambda_0\circ j \;=\; da_\alpha,$$

where j is a complex structure on $\dot{S}$ which equals the standard structure i on $[0,+\infty)\times S^1$ (i.e., near the punctures). We want to turn the maps $\tilde{u}_\alpha = (a_\alpha, u_\alpha) : \dot{S}\to\mathbb{R}\times M$ into $\tilde{J}_0$-holomorphic curves for a suitable almost-complex structure $\tilde{J}_0$ on $\mathbb{R}\times M$. Recall that the contact structure is given by

$$\ker\lambda_\delta \;=\; \text{Span}\{\eta_1,\eta_2\} \;=\; \text{Span}\Big\{\frac{\partial}{\partial r}, -\gamma_2(r)\frac{\partial}{\partial\theta} + \gamma_1(r)\frac{\partial}{\partial\phi}\Big\}.$$

We define complex structures $J_\delta : \ker\lambda_\delta\to\ker\lambda_\delta$ by

$$J_\delta(\theta,r,\phi)\Big(-\gamma_2(r)\frac{\partial}{\partial\theta} + \gamma_1(r)\frac{\partial}{\partial\phi}\Big) := -\frac{1}{h(r)}\,\frac{\partial}{\partial r} \tag{2.7}$$

*A puncture p_j is called *positive* for the curve (a_α, u_α) if $\lim_{z\to p_j} a_\alpha(z) = +\infty$.

and

$$J_\delta(\theta, r, \phi)\frac{\partial}{\partial r} := h(r)\Big(-\gamma_2(r)\frac{\partial}{\partial \theta} + \gamma_1(r)\frac{\partial}{\partial \phi}\Big),$$

where $h : (0,1] \to \mathbb{R}\backslash\{0\}$ are suitable smooth functions. Also recall that γ_1, γ_2 depend on δ away from the binding orbit. We want J_δ to be compatible with $\mathrm{d}\lambda_\delta$, that is, we want

$$d\lambda_\delta(\eta_1, J\eta_1) = h(r)\,\mu(r) > 0 \qquad \text{and} \qquad d\lambda_\delta(\eta_2, J\eta_2) = \frac{\mu(r)}{h(r)} > 0$$

so that $h(r) > 0$. We also demand that J_δ extends smoothly over the binding $\{r = 0\}$. Expressing the vectors η_1 and η_2 in Cartesian coordinates, we have

$$\eta_1 = \frac{1}{r}\Big(x\,\frac{\partial}{\partial x} + y\,\frac{\partial}{\partial y}\Big)$$

and

$$\eta_2 = -\gamma_2(r)\frac{\partial}{\partial \theta} + \gamma_1(r)\,x\,\frac{\partial}{\partial y} - \gamma_1(r)\,y\,\frac{\partial}{\partial x}.$$

We introduce the following generators of the contact structure:

$$\begin{aligned}\varepsilon_1 &:= \gamma_1(r)\frac{\partial}{\partial y} - \frac{x\gamma_2(r)}{r^2}\frac{\partial}{\partial \theta} \\ &= \frac{y\gamma_1(r)}{r}\eta_1 + \frac{x}{r^2}\eta_2\end{aligned}$$

and

$$\begin{aligned}\varepsilon_2 &:= \gamma_1(r)\frac{\partial}{\partial x} + \frac{y\gamma_2(r)}{r^2}\frac{\partial}{\partial \theta} \\ &= \frac{x\gamma_1(r)}{r}\eta_1 - \frac{y}{r^2}\eta_2.\end{aligned}$$

We compute from this

$$\eta_1 = \frac{1}{r\gamma_1(r)}(y\,\varepsilon_1 + x\,\varepsilon_2), \qquad \eta_2 = x\,\varepsilon_1 - y\,\varepsilon_2.$$

Now

$$\begin{aligned}J_\delta\varepsilon_1 &= \frac{y\gamma_1(r)h(r)}{r}\eta_2 - \frac{x}{r^2h(r)}\eta_1 \\ &= \Big(\frac{1}{r}xy\gamma_1(r)h(r) - \frac{xy}{r^3h(r)\gamma_1(r)}\Big)\,\varepsilon_1 \\ &\quad - \Big(\frac{1}{r}y^2\gamma_1(r)h(r) + \frac{x^2}{r^3\gamma_1(r)h(r)}\Big)\,\varepsilon_2\end{aligned}$$

and

$$\begin{aligned}J_\delta\varepsilon_2 &= \frac{x\gamma_1(r)h(r)}{r}\eta_2 + \frac{y}{r^2h(r)}\eta_1 \\ &= \Big(-\frac{1}{r}xy\gamma_1(r)h(r) + \frac{xy}{r^3h(r)\gamma_1(r)}\Big)\,\varepsilon_2 \\ &\quad + \Big(\frac{1}{r}x^2\gamma_1(r)h(r) + \frac{y^2}{r^3\gamma_1(r)h(r)}\Big)\,\varepsilon_1.\end{aligned}$$

Inserting $x = r\cos\phi$, $y = r\sin\phi$, and demanding that the limit is ϕ-independent as $r \to 0$, we arrive at the condition that $r\,h(r)\,\gamma_1(r) \equiv \pm 1$ for small r. Recalling that we need $h > 0$, we obtain

$$h(r) = \frac{1}{r\,\gamma_1(r)} \quad \text{for small } r.$$

As usual, we continue J_δ to an almost-complex structure $\tilde{J}_\delta$ on $\mathbb{R}\times M$ by setting

$$\tilde{J}_\delta(\theta, r, \phi)\frac{\partial}{\partial\tau} := X_\delta(\theta, r, \phi),$$

where τ denotes the coordinate in the $\mathbb{R}$-direction. We emphasize that $\tilde{J}_\delta$ also makes sense for $\delta = 0$. We now arrange $r(s)$ in (2.6) such that the Giroux leaves $\tilde{u}_\alpha = (a_\alpha, u_\alpha)$ become $\tilde{J}_0$-holomorphic curves.* We compute for $r \le 1 - \varepsilon_0$

$$\begin{aligned}\partial_s\tilde{u}_\alpha + \tilde{J}_0(u_\alpha)\partial_t\tilde{u}_\alpha &= \gamma_1(r)\frac{\partial}{\partial\tau} + r'\frac{\partial}{\partial r} + \tilde{J}_0(u_\alpha)\Big(\frac{\partial}{\partial\theta}\Big) \\ &= \gamma_1(r)\frac{\partial}{\partial\tau} + r'\frac{\partial}{\partial r} + \tilde{J}_0(u_\alpha)\big(\gamma_1(r)\,X_\delta(u_\alpha)\big) \\ &\qquad + \tilde{J}_0(u_\alpha)\Big(\frac{\partial}{\partial\theta} - \gamma_1(r)X_\delta(u_\alpha)\Big) \\ &= r'\frac{\partial}{\partial r} + \tilde{J}_0(u_\alpha)\Big(\frac{\gamma_1'(r)}{\mu(r)}\Big(\gamma_1(r)\frac{\partial}{\partial\phi} - \gamma_2(r)\frac{\partial}{\partial\theta}\Big)\Big) \\ &= \Big(r' - \frac{\gamma_1'(r)}{\mu(r)h(r)}\Big)\frac{\partial}{\partial r},\end{aligned}$$

hence the Giroux leaves satisfy the equation if we choose r to be a solution of the ordinary differential equation

$$r'(s) = \frac{\gamma_1'(r(s))}{\mu(r(s))\,h(r(s))}.$$

*The calculation shows that we can make them J_δ-holomorphic for all $\delta \ge 0$ near the binding.

Note that $r'(s) < 0$. We also choose $h(r) \equiv 1$ for $r \geq 1 - \varepsilon_0$. We continue the almost-complex structures $J_\delta : \ker \lambda_\delta \to \ker \lambda_\delta$ (which were only defined near the binding) smoothly to all of M. Away from the binding we have $X_\delta = \partial/\partial\phi$, and we extend J_δ as before to $T(\mathbb{R} \times M)$. Away from the binding, if $\delta = 0$, we have that $\ker \lambda_0$ coincides with the tangent spaces of the pages of the open book decomposition. Because a_α is constant away from the binding, the solutions $\tilde{u}_\alpha$ which we constructed near the binding fit together smoothly with the pages of the open book decomposition and solve the holomorphic curve equation for the almost-complex structure J_0. □

Remark 2.7

Near the binding orbit the function $r(s)$ satisfies a differential equation of the form

$$r'(s) = \Lambda\big(r(s)\big)\, r(s) := \frac{\gamma_1'(r(s))\gamma_1(r(s))}{\mu(r(s))}\, r(s),$$

and

$$\lim_{r \to 0} \Lambda(r) = \frac{\gamma_1''(0)}{\gamma_2''(0)} =: \kappa.$$

Writing $r(s) = c(s)e^{\kappa s}$, the function $c(s)$ satisfies $c'(s) = (\Lambda(r(s)) - \kappa)c(s)$, and hence it is a decreasing function which converges to a constant as $s \to +\infty$.

We return to Examples 2.2 and 2.3, and we compute $r(s)$ for large s. The differential equation in the case of Example 2.3 for large s is

$$r'(s) = \frac{\gamma_1'(r(s))\gamma_1(r(s))}{\mu(r(s))} r(s) = -kT\big(1 - r^2(s)\big) r(s),$$

so that

$$r(s) = \frac{1}{\sqrt{1 + c\, e^{2kTs}}},$$

where c is a constant. In Example 2.2, the differential equation reads

$$r'(s) = \frac{\gamma_1'(r(s))\gamma_1(r(s))}{\mu(r(s))} r(s) = -\frac{kT}{1 - r^2(s)} r(s),$$

and solutions satisfy

$$r(s) = c\, e^{-kT\, s}\, e^{(1/2) r^2(s)}.$$

2.2. Functional analytic setup and the implicit function theorem

In the following theorem we prove the existence of a smooth family of solutions near a given solution. In Proposition 2.6, we constructed a finite energy foliation for the data

(λ_0, J_0) with vanishing harmonic form. The form λ_0, however, is only a confoliation form. We produce solutions for the perturbed data $(\lambda_\delta, J_\delta)$, and harmonic forms appear if the surface S is not a sphere. The key result is an application of the implicit function theorem in a suitable setting.

THEOREM 2.8

Assume one of the following.

(1) *Let* $(a_0, u_0) : \dot{S} \to \mathbb{R} \times M$ *be one of the* $\tilde{J}_0$*-holomorphic curves described in Proposition 2.6 with complex structure* j_0 *on* S *(we refer to such* u_0 *as a* Giroux leaf*) and confoliation form* λ_0.

(2) *Let* $(\dot{S}, j_0, a_0, u_0, \gamma_0)$ *be a solution of the differential equation (1.1) for some* $d\lambda_0$*-compatible complex structure* $J_0 : \ker \lambda_0 \to \ker \lambda_0$ *which, near the binding orbit, agrees with (2.7), and where* λ_0 *is a contact form which is a local model near the binding. Assume that* u_0 *is an embedding and that it is of the form* $u_0 = \phi_g(v_0)$*, where* $g : S \to \mathbb{R}$ *is a smooth function,* ϕ *is the flow of the Reeb vector field, and where* $v_0 : \dot{S} \to M$ *is a Giroux leaf as in Proposition 2.6.*

(3) *Let* J_δ *be a smooth family of* $d\lambda_\delta$*-compatible complex structures also agreeing with (2.7) near the binding orbit, where* $(\lambda_\delta)_{-\varepsilon<\delta<+\varepsilon}$*,* $\varepsilon > 0$ *is a smooth family of* 1*-forms which are contact forms for* $\delta \neq 0$ *and local models near the binding. Then there is a smooth family*

$$(S, j_{\delta,\tau}, a_{\delta,\tau}, u_{\delta,\tau}, \gamma_{\delta,\tau}, J_\delta)_{-\varepsilon<\delta,\tau<+\varepsilon}$$

of solutions of (1.1) so that $u_{\delta,\tau}(\dot{S}) \cap u_{\delta,\tau'}(\dot{S}) = \emptyset$ *whenever* $\tau \neq \tau'$*, and each* $u_{\delta,\tau}$ *is an embedding.*

Proof

In both cases we wish to find solutions of (1.1) for the data $(\lambda_\delta, J_\delta)$ of the form

$$u_\delta(z) = \phi_{f_\delta(z)}\big(u_0(z)\big), \qquad a_\delta(z) = b_\delta(z) + a_0(z),$$

where $t \mapsto \phi_t = \phi_t^\delta$ is the flow of the Reeb vector field X_δ of λ_δ and where $b_\delta + i f_\delta : S \to \mathbb{C}$ is a smooth function defined on the unpunctured surface. We derive an equation for the unknown function $b_\delta + i f_\delta$. From now on we suppress the superscript δ in the notation unless for $\delta = 0$. Because of the first equation in (1.1), the complex structure on S is then determined by f (denote it by $j = j_f$) and is given by

$$j_f(z) = \big(\pi_\lambda T u(z)\big)^{-1} \circ J\big(u(z)\big) \circ \pi_\lambda T u(z). \tag{2.8}$$

Note that this is well defined because u is transverse to the Reeb vector field so that $\pi T u(z) : T_z S \to \ker \lambda(u(z))$ is an isomorphism. By the second equation of (1.1), we then have to solve the equation $df \circ j_f + u_0^* \lambda \circ j_f = da + \gamma$ for a, f, γ on $\dot{S}$ which is equivalent to the equation

$$\bar{\partial}_{j_f}(a + if) = u_0^*\lambda \circ j_f - i(u_0^*\lambda) - \gamma - i(\gamma \circ j_f). \tag{2.9}$$

Recall that we are looking for a of the form $a = a_0 + b$, where b is a suitable real-valued function defined on the whole surface S. We obtain the differential equation

$$\bar{\partial}_{j_f}(b + if) = u_0^*\lambda \circ j_f - i(u_0^*\lambda) - \bar{\partial}_{j_f} a_0 - \gamma - i(\gamma \circ j_f), \tag{2.10}$$

and it follows from a straightforward calculation (see the appendix) that all expressions on the right-hand side of (2.10) are bounded near the punctures; in particular, they are contained in the spaces $L^p(T^*S \otimes \mathbb{C})$ for any p. This is what the assumption $\kappa \le -(1/2)$ from Definition 2.1 is needed for. We work in the function space $b + if \in W^{1,p}(S, \mathbb{C})$, where $p > 2$. For any complex structure j on S, the space $L^p(T^*S \otimes \mathbb{C})$ of complex-valued 1-forms of class L^p decomposes into complex linear and complex antilinear forms (with respect to j). We use the notation

$$L^p(T^*S \otimes \mathbb{C}) = L^p(T^*S \otimes \mathbb{C})^{1,0}_j \oplus L^p(T^*S \otimes \mathbb{C})^{0,1}_j.$$

The operator $b + if \mapsto \bar{\partial}_{j_f}(b + if)$ is then a section in the vector bundle

$$L^p(T^*S \otimes \mathbb{C})^{0,1} := \bigcup_{b+if \in W^{1,p}(S,\mathbb{C})} \{b + if\} \times L^p(T^*S \otimes \mathbb{C})^{0,1}_{j_f} \to W^{1,p}(S, \mathbb{C}).$$

This vector bundle is of course trivial, but here are some explicit local trivializations for $f, \ g \in W^{1,p}(S, \mathbb{R})$ sufficiently close to each other:

$$\begin{aligned} \Psi_{fg} : L^p(T^*S \otimes \mathbb{C})^{0,1}_{j_f} &\xrightarrow{\sim} L^p(T^*S \otimes \mathbb{C})^{0,1}_{j_g} \\ \tau &\longmapsto \tau + i(\tau \circ j_g). \end{aligned} \tag{2.11}$$

If we write

$$\tau + i(\tau \circ j_g) = \tau \circ (\mathrm{Id}_{TS} - j_f \circ j_g),$$

we see that Ψ_{fg} is invertible with

$$\Psi_{fg}^{-1}\tau = \tau \circ (\mathrm{Id}_{TS} - j_f \circ j_g)^{-1}.$$

It follows from the Hodge decomposition theorem that every cohomology class $[\sigma] \in H^1(S, \mathbb{R})$ has a unique harmonic representative $\psi_j(\sigma) \in \mathcal{H}^1_j(S)$, where $\mathcal{H}^1_j(S)$ is defined as

$$\mathcal{H}_j^1(S) := \left\{\gamma \in \mathcal{E}^1(S) \,\middle|\, d\gamma = 0,\ d(\gamma \circ j) = 0\right\} \tag{2.12}$$

and where $\mathcal{E}^1(S)$ denotes the space of all (smooth) real-valued 1-forms on S, and we write $\mathcal{E}^{0,1}(S) = \mathcal{E}_j^{0,1}(S)$ for the space of complex antilinear 1-forms on S with respect to j, that is, complex-valued 1-forms σ such that $i\,\sigma + \sigma\, j = 0$. Note that our definition coincides with the set of closed and co-closed 1-forms on S. Moreover, by elliptic regularity, we may also consider Sobolev forms. We identify $H^1(S, \mathbb{R})$ with $\mathbb{R}^{2g}$, and we consider the following parameter-dependent section in the bundle $L^p(T^*S \otimes \mathbb{C})^{0,1} \to W^{1,p}(S, \mathbb{C})$

$$F : W^{1,p}(S, \mathbb{C}) \times \mathbb{R}^{2g} \longrightarrow L^p(T^*S \otimes \mathbb{C})^{0,1} \tag{2.13}$$

$$\begin{aligned} F(b + if, \sigma) := \bar{\partial}_{j_f}(b + if) - u_0^*\lambda \circ j_f + i(u_0^*\lambda) \\ + \bar{\partial}_{j_f} a_0 + \psi_{j_f}(\sigma) + i\big(\psi_{j_f}(\sigma) \circ j_f\big) \end{aligned}$$

with j_f as in (2.8). Recalling that $z \mapsto j_f(z)$ may not be differentiable, we interpret the equation $d(\gamma \circ j_f) = 0$ in the sense of weak derivatives. The solution set of (2.10) is then the zero set of F. We consider the real parameter δ which we dropped from the notation, fixed at the moment. For $g \equiv 0$ and $b + if$ small in the $W^{1,p}$-norm, we consider the composition $\hat{F}(b + if, \sigma) = \Psi_{fg}(F(b + if, \sigma))$. Its linearization in the point $(b + if, \sigma) = (0, \sigma_0)$, where σ_0 is defined by $\psi_{j_0}(\sigma_0) = \gamma_0$ and where $F(0, \sigma_0) = 0$, is

$$D\hat{F}(0, \sigma_0) : W^{1,p}(S, \mathbb{C}) \times \mathbb{R}^{2g} \longrightarrow L^p(T^*S \otimes \mathbb{C})_{j_0}^{0,1}$$

$$D\hat{F}(0, \sigma_0)(\zeta, \sigma) = \bar{\partial}_{j_0}\zeta + \psi_{j_0}(\sigma) + i(\psi_{j_0}(\sigma) \circ j_0) + L\,\zeta,$$

where

$$L : W^{1,p}(S, \mathbb{C}) \to W^{1,p}(T^*S \otimes \mathbb{C})_{j_0}^{0,1} \hookrightarrow L^p(T^*S \otimes \mathbb{C})_{j_0}^{0,1}$$

is the compact linear map

$$L\zeta = -\frac{1}{2}u_0^*\lambda \circ (A\zeta + j_0 A\zeta j_0) + \frac{i}{2}u_0^*\lambda \circ (j_0 A\zeta - A\zeta j_0) + B\zeta + i\,B\zeta\, j_0,$$

where

$$B\zeta = \frac{d}{d\tau}\bigg|_{\tau=0} \psi_{j_{\tau k}}(\sigma_0), \qquad \zeta = h + ik,$$

and where

$$A\zeta = \frac{d}{d\tau}\bigg|_{\tau=0} j_{\tau k} = h\,(\pi_\lambda T u_0)^{-1}[J(u_0)DX_\lambda(u_0) - DX_\lambda(u_0)J(u_0) + DJ(u_0)X_\lambda(u_0)](\pi_\lambda T u_0).$$

The linear term L therefore does not contribute to the Fredholm index of $D\hat{F}(0, \sigma_0)$. We note that the linear map $\zeta \mapsto L\zeta$ only depends on the imaginary part of ζ. We claim that the operator

$$W^{1,p}(S, \mathbb{C}) \times \mathbb{R}^{2g} \longrightarrow L^p(T^*S \otimes \mathbb{C})^{0,1}_{j_0}$$

$$(\zeta, \sigma) \longmapsto \bar{\partial}_{j_0}\zeta + \psi_{j_0}(\sigma) + i(\psi_{j_0}(\sigma) \circ j_0)$$

is a surjective Fredholm operator of index 2. Then we would have $\operatorname{ind}(D\hat{F}(0, \sigma_0)) = 2$ as well. Here is the argument: The Riemann-Roch theorem asserts that the kernel and the cokernel of the Cauchy-Riemann operator $\bar{\partial}_j$ (acting on smooth complex-valued functions on S) are both finite-dimensional and that

$$\dim_{\mathbb{R}} \ker \bar{\partial}_j - \dim_{\mathbb{R}}\big(\mathcal{E}^{0,1}(S)/\operatorname{Im} \bar{\partial}_j\big) = 2 - 2g,$$

where g is the genus of the surface S. The only holomorphic functions on S are the constant functions, hence $\mathcal{E}^{0,1}(S)/\operatorname{Im} \bar{\partial}_j$ has dimension $2g$.

On the other hand, the vector space $\mathcal{H}^1_j(S)$ of all (real-valued) harmonic 1-forms on S also has dimension $2g$ (see [15]). We now consider the linear map

$$\Psi : \mathcal{H}^1_j(S) \longrightarrow \mathcal{E}^{0,1}(S)/\operatorname{Im} \bar{\partial}_j$$

$$\Psi(\gamma) := [\gamma + i(\gamma \circ j)],$$

where [.] denotes the equivalence classes of (0, 1)-forms. Assume that $\Psi(\gamma) = [0]$, that is, that there is a complex-valued smooth function $f = u + iv$ on S such that $\bar{\partial}_j f = \gamma + i(\gamma \circ j)$. Since γ is a harmonic 1-form, we conclude that $d(dv \circ j) = d(du \circ j) = 0$ (i.e., both u and v are harmonic). Since there are only constant harmonic functions on S we obtain $\gamma = 0$ (i.e., Ψ is injective and also bijective). Hence, (0, 1)-forms $\gamma + i(\gamma \circ j)$ with $\gamma \in \mathcal{H}^1_j(S)$ make up the cokernel of $\bar{\partial}_j : C^\infty(S, \mathbb{C}) \to \mathcal{E}^{0,1}(S)$.

This proves the claim that the operator $D\hat{F}(0, \sigma_0)$ is Fredholm of index 2. We now show that the operator $D\hat{F}(0, \sigma_0)$ is surjective. Using the decomposition

$$L^p(T^*S \otimes \mathbb{C})^{0,1}_{j_0} = R(\bar{\partial}_{j_0}) \oplus \mathcal{H}^1_{j_0}(S)$$

and denoting the corresponding projections by π_1, π_2, we see that it suffices to prove the surjectivity of the operator

$$T : W^{1,p}(S, \mathbb{C}) \to R(\bar{\partial}_{j_0})$$

$$T\zeta := \bar{\partial}_{j_0}\zeta + \pi_1(L\zeta),$$

which is a Fredholm operator of index 2. Assume that $\zeta \in \ker T$. Unless $\zeta \equiv 0$, the set $\{z \in S \mid \zeta(z) = 0\}$ consists of finitely many points by the similarity principle (see [23]), and the local degree of each zero is positive. On the other hand, the sum of all the local degrees has to be zero, hence elements in the kernel of T are nowhere zero. Actually, if $h + ik \in \ker T$, then even k is nowhere zero because $h + c + ik \in \ker T$ for any real constant c since the zero-order term L only depends on the imaginary part of ζ. Therefore,* dim $\ker T \leq 2$, and since the Fredholm index of T equals 2, we actually have dim $\ker T = 2$. This proves the surjectivity of T and also of $D\hat{F}(0, \sigma_0)$ so that the set $\mathcal{M}$ of all pairs $(b + if, \gamma)$ solving the differential equation (2.10) is a 2-dimensional manifold with $T_{(0,\gamma_0)}\mathcal{M} = \ker D\hat{F}(0, \gamma_0)$. If we add a real constant to $b + if$, then we obtain again a solution of (2.10). If we divide $\mathcal{M}$ by this $\mathbb{R}$-action, then we obtain a 1-dimensional family of solutions $(\tilde{u}_\tau)_{-\varepsilon<\tau<\varepsilon}$ with $\tilde{u}_\tau = (a_\tau, u_\tau)$ for which $u_\tau = \phi_{f_\tau}(u_0)$, and the functions f_τ do not vanish at any point. Therefore, we have $u_0(\dot{S}) \cap u_\tau(\dot{S}) = \emptyset$ and also $u_{\tau'}(\dot{S}) \cap u_\tau(\dot{S}) = \emptyset$ if $\tau \neq \tau$. Moreover, the maps u_τ are transverse to the Reeb vector field by construction. □

3. From local foliations to global ones

The aim of this section is to show that a family of solutions produced by the implicit function theorem (see Theorem 2.8) can be enlarged further. For this purpose, a compactness result is needed for which we are setting the stage now.

First, we summarize a result by Siefring which will be used later on.

THEOREM 3.1 ([31, Theorem 2.2])
Let $\tilde{u} \in \mathcal{M}(P, J)$ and $\tilde{v} \in \mathcal{M}(P, J)$, let maps $U, V : [R, \infty) \times S^1 \to C^\infty(P^*\xi)$ be asymptotic representatives of $\tilde{u}$ and $\tilde{v}$, respectively, and assume that $U - V$ does not vanish identically. Then there exists a negative eigenvalue λ of the asymptotic operator $\mathbf{A}_{P,J}$ and an eigenvector e with eigenvalue λ so that

$$U(s, t) - V(s, t) = e^{\lambda s}\big(e(t) + r(s, t)\big),$$

*Indeed, otherwise we would be able to find three linearly independent elements in the kernel $\zeta_1, \zeta_2, \zeta_3$. Because $\mathbb{C}$ has real dimension 2 we can find real numbers $\alpha_1, \alpha_2, \alpha_3$, not all simultaneously zero, and a point $z \in S$ such that $\sum_{j=1}^3 \alpha_j\zeta_j(z) = 0$. Then $\zeta = \sum_{j=1}^3 \alpha_j\zeta_j$ is in the kernel of T and $\zeta(z) = 0$, which is a contradiction.

where the map r satisfies for every $(i, j) \in \mathbb{N}^2$ a decay estimate of the form

$$|\nabla_s^i \nabla_t^j r(s,t)| \leq M_{ij} e^{-ds}$$

with M_{ij} and d positive constants.

Our situation is less general than in [31], so we will explain the notation in the context of this paper. The setup is a manifold M with contact form λ and contact structure $\xi = \ker \lambda$. Consider a periodic orbit $\bar{P}$ of the Reeb vector field X_λ with period T, and we may assume here that T is its minimal period. We introduce $P(t) := \bar{P}(Tt/2\pi)$ such that $P(0) = P(2\pi)$. If $J : \xi \to \xi$ is a $d\lambda$-compatible complex structure, then the set of all $\tilde{J}$-holomorphic half-cylinders

$$\tilde{u} = (a, u) : [R, \infty) \times S^1 \to \mathbb{R} \times M, \qquad S^1 = \mathbb{R}/2\pi\mathbb{Z}$$

for which $|a(s,t) - Ts/2\pi|$ and $|u(s,t) - P(t)|$ decay at some exponential rate (in local coordinates near the orbit $P(S^1)$) is denoted by $\mathcal{M}(P, J)$. Note that it is assumed here that the domain $[R, \infty) \times S^1$ is endowed with the standard complex structure. A smooth map $U : [R, \infty) \times S^1 \to P^*\xi$ for which $U(s,t) \in \xi_{P(t)}$ is called an *asymptotic representative of $\tilde{u}$* if there is a proper embedding $\psi : [R, \infty) \times S^1 \to \mathbb{R} \times S^1$ asymptotic to the identity so that

$$\tilde{u}\big(\psi(s,t)\big) = \big(Ts/2\pi, \exp_{P(t)} U(s,t)\big), \quad \forall\, (s,t) \in [R, \infty) \times S^1$$

(exp is the exponential map corresponding to some metric on M, e.g., the one induced by λ and J). Every $\tilde{u} \in \mathcal{M}(P, J)$ has an asymptotic representative (see [31]). The asymptotic operator $\mathbf{A}_{P,J}$ is defined as follows:

$$(\mathbf{A}_{P,J} h)(t) := -\frac{T}{2\pi} J\big(P(t)\big)\Big(\frac{d}{ds}\Big|_{s=0} D\phi_{-s}(\phi_s(P(t)))h(\phi_s(P(t)))\Big),$$

where ϕ_s is the flow of the Reeb vector field and where h is a section in $P^*\xi \to S^1$. Because the Reeb flow preserves the splitting $TM = \mathbb{R} X_\lambda \oplus \xi$ we have also $(\mathbf{A}_{P,J} h)(t) \in \xi_{P(t)}$.

We compute the asymptotic operator $\mathbf{A}_{P,J}$ for the binding orbit

$$\bar{P}(t) = \Big(\frac{t}{\gamma_1(0)}, 0, 0\Big) \in S^1 \times \mathbb{R}^2.$$

Recall that the above periodic orbit has minimal period $T = 2\pi\gamma_1(0)$. Using

$$\phi_s\big(P(t)\big) = \phi_s(t, 0, 0) = \Big(t + \frac{s}{\gamma_1(0)}, 0, 0\Big),$$

formula (2.2) for the linearization of the Reeb flow with $h(t) = (0, \zeta(t), \eta(t))$, and the fact that $J(t,0,0)\frac{\partial}{\partial x} = \frac{\partial}{\partial y}$ and $J(t,0,0)\frac{\partial}{\partial y} = -\frac{\partial}{\partial x}$, we compute

$$
\begin{aligned}
(\mathbf{A}_{P,J}h)(t) &= -\gamma_1(0)J(t,0,0)\left(\frac{h'(t)}{\gamma_1(0)} + \begin{pmatrix} 0 & \beta(0) \\ -\beta(0) & 0 \end{pmatrix}\begin{pmatrix} \zeta(t) \\ \eta(t) \end{pmatrix}\right) \\
&= -J_0\, h'(t) - \gamma_1(0)\beta(0)\, h(t) \\
&= -J_0\, h'(t) + \kappa\, h(t),
\end{aligned}
$$

where $J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\kappa = \gamma_1''(0)/\gamma_2''(0) \in (-1, 0)$. Hence $\lambda \in \sigma(\mathbf{A}_{P,J})$ precisely if

$$
h'(t) = (\lambda - \kappa)\, J_0\, h(t) \qquad \text{and} \qquad h(2\pi) = h(0)
$$

(i.e., $\sigma(\mathbf{A}_{P,J}) = \{\kappa + l \mid l \in \mathbb{Z}\}$), and the largest negative eigenvalue is given by κ. The corresponding eigenspace consists of all constant vectors $h(t) \equiv const \in \mathbb{R}^2$. The eigenspace for the eigenvalues $\kappa + l$ consists of all

$$
h(t) = e^{J_0 l\, t} h_0, \quad \text{with } h_0 \in \mathbb{R}^2.
$$

THEOREM 3.2 (Compactness)
Let λ be a contact form on M which is a local model near the binding (of the Giroux leaf v_0), and let $J : \ker\lambda \to \ker\lambda$ be a $d\lambda$-compatible complex structure. Consider a smooth family of solutions $(S, j_\tau, a_\tau, u_\tau, \gamma_\tau, J)_{0\le\tau<\tau_0}$ to equation (1.1) satisfying the following conditions.

- *We have $u_\tau = \phi_{f_\tau}(v_0)$, where $v_0 : \dot{S} \to M$ is a Giroux leaf as in Proposition 2.6 and where $f_\tau : S \to \mathbb{R}$ are suitable smooth functions.*
- *For any $0 \le \tau < \tau_0$, there is $\delta > 0$ such that*

$$
u_\tau(\dot{S}) \cap u_{\tau'}(\dot{S}) = \emptyset \quad \textit{whenever } 0 < |\tau - \tau'| < \delta.
$$

- *Assume that u_0 and u_τ never have identical images whenever $0 < \tau < \tau_0$.*

Then the functions f_τ converge uniformly with all derivatives to a smooth function $f_{\tau_0} : S \to \mathbb{R}$ as $\tau \nearrow \tau_0$. The harmonic 1-forms γ_τ also converge in $C^\infty(S)$ to a 1-form γ_{τ_0} which is harmonic with respect to the complex structure j_{τ_0} on $\dot{S}$ given by

$$
j_{\tau_0}(z) := \big(\pi_\lambda T u_{\tau_0}(z)\big)^{-1} \circ J\big(u_{\tau_0}(z)\big) \circ \pi_\lambda T u_{\tau_0}(z),
$$

where $u_{\tau_0} := \phi_{f_{\tau_0}}(v_0)$. Moreover, we can find a smooth function a_{τ_0} on $\dot{S}$ so that $(S, j_{\tau_0}, a_{\tau_0}, u_{\tau_0}, \gamma_{\tau_0}, J)$ solves the differential equation (1.1).

Remark 3.3
We may assume without loss of generality that $v_0 \equiv u_0$ and $f_0 \equiv 0$. If $z \in \dot{S}$, then we denote by $T(z) > 0$ the positive return time of the point $u_0(z)$; that is, we have

$$T(z) := \inf \left\{ T > 0 \,\middle|\, \phi_T(u_0(z)) \in u_0(\dot{S}) \right\} < +\infty.$$

We claim that the return time $z \mapsto T(z)$ extends continuously over the punctures of the surface, and that therefore there is an upper bound

$$T := \sup_{z \in \dot{S}} T(z) < \infty.$$

Using (2.1) and (2.6), we note that, asymptotically near the punctures, $\phi_T(u_0(s,t)) \in S^1 \times \mathbb{R}^2$ has the following structure:

$$\phi_T(u_0(s,t)) = \big(t + \alpha(r(s))T,\ r(s)\exp[i(\alpha_0 + \beta(r(s))T)]\big),$$

where $r(s)$ is a strictly decreasing function, α_0 is some constant, and $\alpha(r)$, $\beta(r)$ are suitable functions for which the limits $\lim_{r\to 0} \beta(r)$ and $\lim_{r\to 0} \alpha(r)$ exist and are not zero. Hence, if $T = T(u_0(s,t))$ is the positive return time at the point $u_0(s,t)$, then

$$T\big(u_0(s,t)\big) = \frac{2\pi}{|\beta(r(s))|},$$

and therefore the limit for $s \to +\infty$ exists.

The remainder of this section is devoted to the proof of Theorem 3.2. We recall that the functions a_τ and f_τ satisfy the Cauchy-Riemann type equation (2.9) which is

$$\bar{\partial}_{j_\tau}(a_\tau + i f_\tau) = u_0^*\lambda \circ j_\tau - i(u_0^*\lambda) - \gamma_\tau - i(\gamma_\tau \circ j_\tau),$$

where the complex structure j_τ is given by (2.8) or

$$\begin{aligned} j_\tau(z) = {} & \big(\pi_\lambda T u_0(z)\big)^{-1} \big(T\phi_{f_\tau(z)}(u_0(z))\big)^{-1} \\ & \cdot J\big(\phi_{f_\tau(z)}(u_0(z))\big)\, T\phi_{f_\tau(z)}\big(u_0(z)\big)\pi_\lambda T u_0(z), \end{aligned}$$

and that γ_τ is a closed 1-form on S with $d(\gamma_\tau \circ j_\tau) = 0$.

The following L^∞-bound is the crucial ingredient for the compactness result. We claim that

$$\sup_{0 \le \tau < \tau_0} \|f_\tau\|_{L^\infty(\dot{S})} \le T. \tag{3.1}$$

Restricting any of the solutions to a simply connected subset $U \subset \dot{S}$, we can write $\gamma_\tau = dh_\tau$ for a suitable function $h_\tau : U \to \mathbb{R}$, and the maps

$$\tilde{u}_\tau : U \to \mathbb{R} \times M, \qquad \tilde{u}_\tau = (a_\tau + h_\tau, u_\tau)$$

are $\tilde{J}$-holomorphic curves. If two such curves $\tilde{u}_\tau$ and $\tilde{u}_{\tau'}$ have an isolated intersection, then the corresponding intersection number is positive (see [28], [5], or [27] for positivity of (self-)intersections for holomorphic curves). We claim that

$$u_0(\dot{S}) \cap u_\tau(\dot{S}) = \emptyset, \quad \forall\ 0 < \tau < \tau_0,$$

and not just for small τ as assumed. If we can show this, then (3.1) follows. Indeed, for any $z \in \dot{S}$, the function $\tau \mapsto f_\tau(z)$ is strictly increasing from $f_0(z) = 0$, and equality $f_\tau(z) = T(z)$ would imply that $u_\tau(z) \in u_0(\dot{S})$. Arguing indirectly, we assume that the set

$$\mathcal{O} := \left\{\tau \in (0, \tau_0) \,\middle|\, u_\tau(\dot{S}) \cap u_0(\dot{S}) \neq \emptyset\right\}$$

is not empty. We denote its infimum by $\tilde{\tau}$, which must be a positive number since $u_\tau(\dot{S}) \cap u_0(\dot{S}) = \emptyset$ for all sufficiently small $\tau > 0$.

We first prove that the above set is open, which implies that $u_{\tilde{\tau}}$ and u_0 cannot intersect. If $u_\tau(p) = u_0(q)$ for suitable points $p, q \in \dot{S}$, then we consider locally near these points the corresponding holomorphic curves $\tilde{u}_\tau$ and $\tilde{u}_0$. Adding some constant to the $\mathbb{R}$-component of one of them, we may assume that $\tilde{u}_\tau(p) = \tilde{u}_0(q)$. If this intersection point is not isolated, then p and q have open neighborhoods U and V, respectively, on which the holomorphic curves $\tilde{u}_\tau$ and $\tilde{u}_0$ agree. This implies that the set of all points $p \in \dot{S}$ such that $\tilde{u}_\tau(p)$ is a non-isolated intersection point between $\tilde{u}_\tau$ and $\tilde{u}_0$, is open and closed, that is, it is either empty or all of $\dot{S}$. Since we assumed that each set $u_\tau(\dot{S})$, $\tau > 0$ is different from $u_0(\dot{S})$, we conclude that if u_τ and u_0 intersect, then the intersection point of the corresponding holomorphic curves $\tilde{u}_\tau$ and $\tilde{u}_0$ must be isolated. But on the other hand, this implies that $u_{\tau'}$ and u_0 would also intersect for all τ' sufficiently close to τ by positivity of the intersection number showing that the set $\mathcal{O}$ is open.

We conclude from the above that we have a sequence $\tau_k \searrow \tilde{\tau}$ and points $p_k, q_k \in \dot{S}$ such that $u_{\tau_k}(p_k) = u_0(q_k)$. Passing to a suitable subsequence, we may assume convergence of the sequences $(p_k)_{k\in\mathbb{N}}$ and $(q_k)_{k\in\mathbb{N}}$ to points $p, q \in S$. Because of $u_{\tilde{\tau}}(\dot{S}) \cap u_0(\dot{S}) = \emptyset$ the points p, q must be punctures, and they have to be equal $z_0 = p = q \in S \backslash \dot{S}$. The reason for this is the following. The maps u_{τ_k}, u_0 are asymptotic near the punctures to a disjoint union of finitely many periodic Reeb orbits which are not iterates of other periodic orbits. Also, different punctures always correspond to different periodic orbits. This follows from Giroux's result and our constructions in Section 2 of this paper.

We now derive a contradiction using Siefring's result. The harmonic forms γ_{τ_k} in equation (1.1) are defined on all of S. Hence they are exact on some open neighborhood U of the puncture z_0, and $\gamma_{\tau_k} = dh_{\tau_k}$ for suitable functions h_{τ_k} on U and similarly $\gamma_{\tilde{\tau}} = dh_{\tilde{\tau}}$. We may also assume that $j_{\tilde{\tau}}|_U = j_{\tau_k}|_U = j_0$ after changing local coordinates near z_0. Then on the set U, the maps $\tilde{u}_{\tau_k} = (a_{\tau_k} + h_{\tau_k}, u_{\tau_k})$ and $\tilde{u}_0 = (a_0 + h_0, u_0)$ are holomorphic curves with $\tilde{u}_{\tau_k}(p_k) = \tilde{u}_0(q_k)$ while the images of $\tilde{u}_{\tilde{\tau}}$ and $\tilde{u}_0$ have empty intersection. Now let

$$U_{\tilde{\tau}},\ U_{\tau_k},\ U_0 : [R, \infty) \times S^1 \to \mathbb{R}^2$$

be asymptotic representatives of the holomorphic curves $\tilde{u}_{\tilde{\tau}}$, $\tilde{u}_{\tau_k}$, $\tilde{u}_0$, respectively. Invoking Theorem 3.1 and our subsequent computation of the asymptotic operator and its spectrum, we obtain the following asymptotic formulas

$$U_\tau(s,t) - U_0(s,t) = e^{\lambda_\tau s}\big(e_\tau(t) + r_\tau(s,t)\big), \qquad \tau = \tilde{\tau}, \tau_k, \qquad s \geq R_\tau, \tag{3.2}$$

where $R_\tau > 0$ is some constant and where $\lambda_\tau < 0$ is some negative eigenvalue of the asymptotic operator $\mathbf{A}_{P,J}$. It is of the form $\lambda_\tau = \kappa + l_\tau$, where l_τ is an integer, $\kappa = \gamma_1''(0)/\gamma_2''(0)$ is not an integer, and where $e_\tau(t) = e^{J_0 l_\tau t} h_\tau$, $h_\tau \in \mathbb{R}^2 \backslash \{0\}$ is an eigenvector corresponding to the eigenvalue $\lambda_\tau = \kappa + l_\tau$. Note that the above formula applies since $U_\tau - U_0$ cannot vanish identically. We will actually show that $l_\tau \equiv 0$. The asymptotic representative U_0 is given by

$$u_0(s,t) = \big(t, r(s)e^{i\alpha_0}\big) = \big(t, U_0(s,t)\big),$$

using equation (2.6), and we recall that $r(s) = c(s)e^{\kappa s}$, where $c(s) \to c_\infty > 0$ as $s \to +\infty$. An asymptotic representative of $\tilde{u}_\tau$, however, is given by an expression such as

$$u_\tau\big(\psi(s,t)\big) = \big(t, U_\tau(s,t)\big),$$

where $\psi : [R,\infty) \times S^1 \to \mathbb{R} \times S^1$ is a proper embedding converging to the identity map as $s \to +\infty$. Writing $(s', t') = \psi(s,t)$, we get using equations (2.1) for the Reeb flow

$$\begin{aligned} U_\tau(s,t) &= c(s')e^{\kappa s'} e^{i(\alpha_0 + \beta(r(s'))f_\tau(s',t'))} \\ &= e^{\kappa s}\big(e_\tau + r_\tau(s,t)\big). \end{aligned}$$

The asymptotic formula for U_τ a priori allows for other decay rates, but κ is the only possible one. Dividing by $e^{\kappa s}$ and passing to the limit $s \to +\infty$, we obtain

$$e_\tau = c_\infty e^{i\alpha_0} e^{i\beta(0) f_\tau(\infty)},$$

where $f_\tau(\infty) = \lim_{s\to+\infty} f_\tau(s,t)$ which is independent of t since f_τ extends continuously over the punctures. Hence the difference $U_\tau - U_0$ has decay rate $\lambda_\tau \equiv \kappa$ as claimed unless the two eigenvectors e_τ and e_0 agree, which is equivalent to

$$f_\tau(\infty) \in \frac{2\pi}{\beta(0)}\mathbb{Z}$$

or $\tau = \tilde{\tau}$ in our case. The maps $U_\tau - U_0$ satisfy a Cauchy-Riemann type equation to which the similarity principle applies so that, for every zero (s,t) of $U_\tau - U_0$, the map $\sigma \mapsto (U_\tau - U_0)(s + \epsilon\cos\sigma, t + \epsilon\sin\sigma)$ has positive degree for small $\epsilon > 0$. The Cauchy-Riemann type equation mentioned above is derived in [31, Section 5.3] as well as in [1, Section 3] in a slightly different context, and also in [21]. If R is sufficiently large, then the map

$$S^1 \to S^1, \qquad t \mapsto W_\tau(R,t) := \frac{U_\tau - U_0}{|U_\tau - U_0|}(R,t)$$

is well defined, and it has degree l_τ because the remainder term $r_\tau(s,t)$ decays exponentially in s. Zeros of $U_\tau - U_0$ contribute in the following way: if $R' < R$ such that $(U_\tau - U_0)(R',t) \neq 0$, then

$$\deg W_\tau(R,\cdot) \;=\; \deg W_\tau(R',\cdot) + \sum_{\{z \mid U_\tau(z) - U_0(z) = 0\}} o(z). \tag{3.3}$$

We know already that $l_\tau = 0$ whenever $\tau \neq \tilde{\tau}$. Arguing indirectly, we assume that $l_{\tilde{\tau}}$ is not zero. It would have to be negative then. Choose then $R' > 0$ so large that $\deg W_{\tilde{\tau}}(R',\cdot) = l_{\tilde{\tau}} < 0$. For τ sufficiently close to $\tilde{\tau}$, we also have $\deg W_\tau(R',\cdot) = l_{\tilde{\tau}}$. On the other hand, we have $\deg W_\tau(R,\cdot) = 0$ for $R > R'$ sufficiently large. Equation (3.3) implies that the map $U_\tau - U_0$ must have zeros in $[R', R] \times S^1$ to account for the difference in degrees, but we know that there are none for $\tau < \tilde{\tau}$. This contradiction shows that $l_{\tilde{\tau}} \neq 0$ is impossible. Choose again $R' > 0$ so large that $\deg W_{\tilde{\tau}}(R',\cdot) = 0$. The degree does not change if we slightly alter τ. In particular, we have $\deg W_\tau(R',\cdot) = 0$ for $\tau > \tilde{\tau}$ close to $\tilde{\tau}$ as well. For $R >> R'$, we have $\deg W_{\tilde{\tau}}(R,\cdot) = 0$, and we recall that

$$(U_{\tau_k} - U_0)(s_k, t_k) = 0, \;\; \tau_k \searrow \tilde{\tau}$$

for a suitable sequence (s_k, t_k) with $s_k \to +\infty$ and that the set of zeros of $U_{\tau_k} - U_0$ is discrete. This, however, contradicts equation (3.3) since the zeros have positive orders. Summarizing, we have shown that the assumption $\mathcal{O} \neq \emptyset$ leads to a contradiction which implies the a priori bound (3.1).

The monotonicity of the functions f_τ in τ and the bound (3.1) imply that the functions f_τ converge pointwise to a measurable function f_{τ_0} as $\tau \nearrow \tau_0$. We also know that $\|f_{\tau_0}\|_{L^\infty(\dot{S})} \leq T$. We then obtain a complex structure j_{τ_0} on $\dot{S}$ by

$$j_{\tau_0}(z) = \big(\pi_\lambda T u_0(z)\big)^{-1} \big(T\phi_{f_{\tau_0}(z)}(u_0(z))\big)^{-1} \\ \times J\big(\phi_{f_{\tau_0}(z)}(u_0(z))\big)\, T\phi_{f_{\tau_0}(z)}\big(u_0(z)\big)\pi_\lambda T u_0(z).$$

By definition, the complex structure j_{τ_0} is also of class L^∞ and $j_\tau(z) \to j_1(z)$ pointwise. Our task is to improve the regularity of the limit f_{τ_0} and the character of the convergence $f_\tau \to f_{\tau_0}$. We also have to establish convergence of the functions a_τ for $\tau \nearrow \tau_0$. The complex structures j_τ are of course all smooth, but the limit j_{τ_0} might only be measurable.

3.1. The Beltrami equation

For the reader's convenience, we briefly summarize a few classical facts from the theory of quasiconformal mappings (see [8], [9]). The punctured surface $\dot{S}$ carries metrics g_τ, also of class L^∞ for $\tau = \tau_0$ and smooth otherwise, so that

$$g_\tau(z)\big(j_\tau(z)v, j_\tau(z)w\big) = g_\tau(z)(v, w), \quad \text{for all } v, w \in T_z\dot{S}.$$

In fact, g_τ is given by

$$g_\tau(z)(v, w) = d\lambda\big(u_\tau(z)\big)\big(\pi_\lambda T u_\tau(z)v, J(u_\tau(z))\pi_\lambda T u_\tau(z)w\big).$$

In the case $\tau = \tau_0$, we replace $\pi_\lambda T u_\tau(z)$ by $T\phi_{f_{\tau_0}(z)}(u_0(z))\pi_\lambda T u_0(z)$. We have $\sup_\tau \|g_\tau\|_{L^\infty(\dot{S})} < \infty$ and $g_\tau \to g_{\tau_0}$ pointwise as $\tau \nearrow \tau_0$. Our considerations about the regularity of the limit are of local nature, so we may replace $\dot{S}$ with a ball $B \subset \mathbb{C}$ centered at the origin. Denoting the metric tensor of g_τ by $(g^\tau_{kl})_{1\leq k,l\leq 2}$, we define the following complex-valued smooth functions:

$$\mu_\tau(z) := \frac{\frac{1}{2}(g^\tau_{11}(z) - g^\tau_{22}(z)) + i\, g^\tau_{12}(z)}{\frac{1}{2}(g^\tau_{11}(z) + g^\tau_{22}(z)) + \sqrt{g^\tau_{11}(z)g^\tau_{22}(z) - (g^\tau_{12}(z))^2}},$$

and we note that

$$\sup_\tau \|\mu_\tau\|_{L^\infty(\dot{S})} < 1$$

and that $\mu_\tau \to \mu_{\tau_0}$ pointwise. We view the functions μ_τ as functions on the whole complex plane by trivially extending them beyond B. Then they are also τ-uniformly bounded in $L^p(\mathbb{C})$ for all $1 \leq p \leq \infty$ and $\mu_\tau \to \mu_{\tau_0}$ in $L^p(\mathbb{C})$ for $1 \leq p < \infty$ by Lebesgue's theorem. If we now solve the Beltrami equation

$$\bar{\partial}\alpha_\tau = \mu_\tau\, \partial\alpha_\tau$$

for $\tau < \tau_0$ so that $\partial\alpha_\tau(0) \neq 0$, then α_τ is a diffeomorphism of the plane onto itself so that

$$g_\tau\big(\alpha_\tau(z)\big)\big(T\alpha_\tau(z)v, T\alpha_\tau(z)w\big) = \lambda_\tau\langle v, w\rangle \quad \text{if } z \in B,$$

where $\langle\, .\,,\, .\,\rangle$ denotes the standard Euclidean scalar product on $\mathbb{R}^2$ and where λ_τ is a positive function. We then get

$$T\alpha_\tau(z) \circ i \;=\; j_\tau(\alpha_\tau(z)) \circ T\alpha_\tau(z), \quad 0 \le \tau < 1 \text{ if } z \in B.$$

For Hölder-continuous μ_τ the map α_τ exists, and it is a C^1-diffeomorphism. This is a classical result by Korn and Lichtenstein [26] (more modern proofs may be found, e.g., in [11] and [13]). In our case, we have smooth solutions α_τ belonging to smooth μ_τ, but we only know that the μ_τ converge pointwise as $\tau \nearrow 1$. On the other hand, we would like to derive a decent notion of convergence for the transformations α_τ. An interesting case for us is the one where μ is only a measurable function. Results in this direction were obtained by Morrey [29], Ahlfors and Bers [9], and also by Bers and Nirenberg [12]. We also refer to [8] by Ahlfors. We now summarize a few results from [9] about the Beltrami equation for measurable μ which we will need later on. The first result concerns the inhomogeneous Beltrami equation

$$\bar\partial u = \mu\, \partial\, u + \sigma,$$

where $u : \mathbb{C} \to \mathbb{C}$, μ is a complex-valued measurable function on $\mathbb{C}$ with

$$\|\mu\|_{L^\infty(\mathbb{C})} < 1$$

and $\sigma \in L^p(\mathbb{C})$ for a suitable $p > 2$ (we explain shortly what values for p are admissible). We consider the following operators acting on smooth functions with compact support in the plane:

$$(Ag)(z) := \frac{1}{2\pi i}\int_{\mathbb{C}} g(\xi)\Big(\frac{1}{\xi - z} - \frac{1}{\xi}\Big)d\xi\, d\bar\xi,$$

$$(\Gamma g)(z) := \frac{1}{2\pi i}\lim_{\varepsilon \searrow 0}\int_{\mathbb{C}\setminus B_\varepsilon(0)} \frac{g(\xi) - g(z)}{(\xi - z)^2}d\xi\, d\bar\xi.$$

Both operators can be extended to continuous operators $L^p(\mathbb{C}) \to L^p(\mathbb{C})$ for all $2 < p < \infty$. They have the following properties:

(1) $\bar\partial(Ag) = A(\bar\partial g) = g$;

(2) $\partial(Ag) = A(\partial g) = \Gamma g$;

(3) $|Ag(z_1) - Ag(z_2)| \le C_p\|g\|_{L^p(\mathbb{C})}|z_2 - z_1|^{1-2/p}$;

(4) $\|\Gamma g\|_{L^p(\mathbb{C})} \le c_p\, \|g\|_{L^p(\mathbb{C})}$ with $c_p \to 1$ as $p \to 2$.

We used here the notation $\overline{\partial} = (1/2)(\partial_s + i\partial_t)$ and $\partial = (1/2)(\partial_s - i\partial_t)$. Properties (1) and (2) above should be understood in the sense of distributions. The proof of (4) involves the Calderón-Zygmund inequality and the Riesz-Thorin convexity theorem (see [25] and [32]). Following [9], we define B_p with $p > 2$ to be the space of all locally integrable functions on the plane which have weak derivatives in $L^p(\mathbb{C})$, vanish in the origin, and which satisfy a global Hölder condition with exponent $1 - \frac{2}{p}$. For $u \in B_p$, we then define a norm by

$$\|u\|_{B_p} := \sup_{z_1 \neq z_2} \frac{|u(z_2) - u(z_1)|}{|z_2 - z_1|^{1-(2/p)}} + \|\partial u\|_{L^p(\mathbb{C})} + \|\overline{\partial} u\|_{L^p(\mathbb{C})}$$

so that B_p becomes a Banach space. We usually choose $p > 2$ such that $c_p \sup_\tau \|\mu_\tau\|_{L^\infty(\mathbb{C})} < 1$, where c_p is the constant from item (4) above.

THEOREM 3.4 ([9, Theorem 1])
Assume that $p > 2$ such that $c_p \sup_\tau \|\mu_\tau\|_{L^\infty(\mathbb{C})} < 1$. If $\sigma \in L^p(\mathbb{C})$, then the equation

$$\overline{\partial} u = \mu\, \partial u + \sigma$$

has a unique solution $u = u_{\mu,\sigma} \in B_p$.

For the existence part of the theorem, one first solves the following fixed-point problem in $L^p(\mathbb{C})$:

$$q = \Gamma(\mu\, q) + \Gamma\sigma.$$

This is possible because the map

$$\begin{aligned} L^p(\mathbb{C}) &\longrightarrow L^p(\mathbb{C}) \\ q &\longmapsto \Gamma(\mu q + \sigma) \end{aligned}$$

is a contraction in view of $c_p \|\mu\|_{L^\infty(\mathbb{C})} < 1$. Then

$$u := A(\mu\, q + \sigma)$$

is the desired solution. The following estimate is also derived in [9]:

$$\|q\|_{L^p(\mathbb{C})} \leq c'_p \|\sigma\|_{L^p(\mathbb{C})} \tag{3.4}$$

with $c'_p = c_p/(1 - c_p\|\mu\|_{L^\infty(\mathbb{C})})$, which follows from

$$\begin{aligned} \|q\|_{L^p(\mathbb{C})} &\leq \|\Gamma(\mu q)\|_{L^p(\mathbb{C})} + \|\Gamma\sigma\|_{L^p(\mathbb{C})} \\ &\leq c_p \|\mu\|_{L^\infty(\mathbb{C})} \|q\|_{L^p(\mathbb{C})} + c_p \|\sigma\|_{L^p(\mathbb{C})}. \end{aligned}$$

Recalling our original situation, we have the following result which shows that there is some sort of conformal mapping for j_1 on the ball B.

THEOREM 3.5 ([9, Theorem 4])
Let $\mu : \mathbb{C} \to \mathbb{C}$ *be an essentially bounded measurable function with* $\mu|_{\mathbb{C}\setminus B} \equiv 0$ *and* $p > 2$ *such that* $c_p\|\mu\|_{L^\infty(\mathbb{C})} < 1$. *Then there is a unique map* $\alpha : \mathbb{C} \to \mathbb{C}$ *with* $\alpha(0) = 0$ *such that*

$$\bar{\partial}\alpha = \mu\partial\alpha$$

in the sense of distributions with $\partial\alpha - 1 \in L^p(\mathbb{C})$.

The desired map α is given by

$$\alpha(z) = z + u(z),$$

where $u \in B_p$ solves the equation $\bar{\partial}u = \mu\partial u + \mu$. In particular, $\alpha \in W^{1,p}(B)$. Lemma 8 in [9] states that $\alpha : \mathbb{C} \to \mathbb{C}$ is a homeomorphism. We can apply the theorem to all the μ_τ, $0 < \tau \leq 1$ and obtain smooth μ_τ-conformal mappings α_τ together with the associated maps u_τ.

LEMMA 3.6
Let $\mu_n : \mathbb{C} \to \mathbb{C}$ *be a sequence of measurable functions so that* $\mu_n|_{\mathbb{C}\setminus B} \equiv 0$ *and* $\sup_n \|\mu_n\|_{L^\infty(\mathbb{C})} < 1$. *Assume that* $\mu_n \to \mu$ *pointwise almost everywhere. Then the corresponding quasi-conformal mappings* α_n, α *as in Theorem 3.5 satisfy*

$$\|\alpha_n - \alpha\|_{W^{1,p}(B)} \longrightarrow 0$$

as $n \to \infty$ *for any* $p > 2$ *such that* $c_p \sup_n \|\mu_n\|_{L^\infty(\mathbb{C})} < 1$ *and any compact set* $B \subset \mathbb{C}$.

Proof
We first estimate with $g \in L^p(\mathbb{C})$ and $z \neq 0$

$$\begin{aligned} |Ag(z)| &= \frac{1}{2\pi}\left| \int_{\mathbb{C}} g(\xi)\frac{z}{\xi(\xi - z)}\right| d\xi\, d\bar{\xi} \\ &\leq \frac{|z|}{2\pi}\|g\|_{L^p(\mathbb{C})}\left\| \frac{1}{\xi(\xi - z)}\right\|_{L^{\frac{p}{p-1}}(\mathbb{C})} \\ &\leq C_p\|g\|_{L^p(\mathbb{C})}\,|z|^{1-\frac{2}{p}}, \end{aligned} \tag{3.5}$$

where the last estimate holds in view of

$$\int_{\mathbb{C}} |\xi(\xi - z)|^{-p/(p-1)}\, d\xi\, d\bar{\xi} \overset{\zeta = z^{-1}\xi}{=} \int_{\mathbb{C}} |z^2\zeta^2 - z^2\zeta|^{-p/(p-1)} |z|^2\, d\zeta\, d\bar{\zeta}$$
$$= |z|^{2-2p/(p-1)} \underbrace{\int_{\mathbb{C}} |\zeta(\zeta - 1)|^{-p/(p-1)} d\zeta\, d\bar{\zeta}}_{2\pi C_p}.$$

If q solves $q = \Gamma(\mu q + \mu)$, then

$$\bar{\partial}(\alpha_n - \alpha) = \mu_n\, \partial(\alpha_n - \alpha) - \mu\, \partial\alpha + \mu_n\, \partial\alpha$$
$$= \mu_n\, \partial(\alpha_n - \alpha) + \mu_n - \mu + (\mu_n - \mu)\Gamma(\mu q + \mu),$$

that is, the difference $\alpha_n - \alpha$ again satisfies an inhomogeneous Beltrami equation. By Theorem 3.4, we have

$$\alpha_n - \alpha = A(\mu_n q_n + \lambda_n),$$

where $\lambda_n = \mu_n - \mu + (\mu_n - \mu)\Gamma(\mu q + \mu)$ and where $q_n \in L^p(\mathbb{C})$ solves $q_n = \Gamma(\mu_n q_n + \lambda_n)$. Combining this with (3.5) and (3.4), we obtain

$$|\alpha_n(z) - \alpha(z)| \le C_p\, \|\mu_n q_n + \lambda_n\|_{L^p(\mathbb{C})}\, |z|^{1-2/p}$$
$$\le (C_p \sup_n \|\mu_n\|_{L^\infty(\mathbb{C})} \cdot c_p' \|\lambda_n\|_{L^p(\mathbb{C})} + C_p\, \|\lambda_n\|_{L^p(\mathbb{C})})\, |z|^{1-2/p}. \tag{3.6}$$

Since $\|\mu_n - \mu\|_{L^p(\mathbb{C})} \to 0$ and $\|(\mu_n - \mu)\Gamma(\mu q + \mu)\|_{L^p(\mathbb{C})} \to 0$ by Lebesgue's theorem we also have $\|\lambda_n\|_{L^p(\mathbb{C})} \to 0$ and therefore $\alpha_n \to \alpha$ uniformly on compact sets. Since $\bar{\partial}(\alpha_n - \alpha) = \mu_n\, \partial(\alpha_n - \alpha) + \lambda_n$ and $\alpha_n - \alpha = A(\mu_n q_n + \lambda_n)$, we verify that

$$\partial(\alpha_n - \alpha) = \Gamma(\mu_n q_n + \lambda_n) = q_n$$

and

$$\bar{\partial}(\alpha_n - \alpha) = \mu_n q_n + \lambda_n.$$

Invoking (3.4) once again, we see that both $\|\partial(\alpha_n - \alpha)\|_{L^p(\mathbb{C})}$ and $\|\bar{\partial}(\alpha_n - \alpha)\|_{L^p(\mathbb{C})}$ can be bounded from above by a constant times $\|\lambda_n\|_{L^p(\mathbb{C})}$ which converges to zero. □

We also need some facts concerning the classical case where $\mu \in C^{k,\alpha}(B_R(0))$, $B_R(0) = \{z \in \mathbb{C} \mid |z| < R\}$ which are not spelled out explicitly in either [11] or [13], but which easily follow from the constructions carried out there.

THEOREM 3.7
Let $\mu, \gamma, \delta \in C^{k,\alpha}(B_{R'}(0))$ *with* $0 < \alpha < 1$ *and* $\sup_{B_{R'}(0)} |\mu| < 1$. *Then for sufficiently small* $0 < R \leq R'$, *there is a unique solution* $w \in C^{k+1,\alpha}(B_R(0))$ *to the equation*

$$\overline{\partial} w(z) = \mu(z)\partial w(z) + \gamma(z)w(z) + \delta(z)$$

with $w(0) = 0$ *and* $\partial w(0) = 1$. *If* w_1, w_2 *solve the above equation with coefficient functions* $\mu_l, \gamma_l, \delta_l, l = 1, 2$, *then there is a constant* $c = c(\alpha, R, \|w_2\|_{C^{k,\alpha}(B_R(0))}, k) > 0$ *such that, for all* $w_1 \in C^{k+1,\alpha}(B_R(0))$, *we have*

$$\|w_2 - w_1\|_{C^{k+1,\alpha}(B_R(0))} \leq c\,(\|\delta_2 - \delta_1\|_{C^{k,\alpha}(B_R(0))} + \|\mu_2 - \mu_1\|_{C^{k,\alpha}(B_R(0))} + \|\gamma_2 - \gamma_1\|_{C^{k,\alpha}(B_R(0))}).$$

Sketch of the proof
The existence proof is a slight generalization of the Korn-Lichtenstein result (see also [11] or [13]). What we are looking for is the estimate. We define the following operator

$$(Tw)(z) := A(\mu\partial w + \gamma w)(z) - z\,\Gamma(\mu\partial w + \gamma w)(0)$$

and the function

$$g(z) := (A\delta)(z) - z\,(\Gamma\delta)(0) + z.$$

A solution to the problem

$$w(z) = (Tw)(z) + g(z)$$

also solves the equation $\overline{\partial} w(z) = \mu(z)\partial w(z) + \gamma(z)w(z) + \delta(z)$ with $w(0) = 0$ and $\partial w(0) = 1$. In [11, Lecture 4] it is shown that T defines a bounded linear operator

$$T : C^{1,\alpha}\big(B_R(0)\big) \longrightarrow C^{1,\alpha}\big(B_R(0)\big)$$

with

$$\|T\| \leq \text{const} \cdot R^\alpha = \theta, \quad \theta < 1 \text{ for small } R > 0,$$

so that the series $g + \sum_{k=1}^{\infty} T^k g$ converges and the limit w satisfies $w = Tw + g$. Another useful fact is the following. Assume that T_1, T_2 are operators as above with coefficient functions μ_1, γ_1 and μ_2, γ_2, respectively. Then

$$\|T_2 - T_1\| \leq c\,(\|\mu_2 - \mu_1\|_{C^{0,\alpha}(B_R(0))} + \|\gamma_2 - \gamma_1\|_{C^{0,\alpha}(B_R(0))})$$

for a suitable constant $c > 0$ depending on α and R. This is only implicitly proved in [11], so we sketch the proof of this inequality. We have

$$\begin{aligned}(T_2 - T_1)h(z) = A\big((\mu_2 - \mu_1)\partial h + (\gamma_2 - \gamma_1)h\big)(z) \\ - z\,\Gamma\big((\mu_2 - \mu_1)\partial h + (\gamma_2 - \gamma_1)h\big)(0),\end{aligned}$$

$$\begin{aligned}\partial\big((T_2 - T_1)h\big)(z) = \Gamma\big((\mu_2 - \mu_1)\partial h + (\gamma_2 - \gamma_1)h\big)(z) \\ - \Gamma\big((\mu_2 - \mu_1)\partial h + (\gamma_2 - \gamma_1)h\big)(0),\end{aligned}$$

and

$$\bar\partial\big((T_2 - T_1)h\big)(z) = (\mu_2 - \mu_1)(z)\partial h(z) + (\gamma_2 - \gamma_1)(z)h(z).$$

We need [13, (21)–(24)]. Adapted to our notation, they look as follows with $z, z_1, z_2 \in B_R(0)$:

$$\begin{aligned}|(Ah)(z)| &\le 4R\|h\|_{C^0(B_R(0))} \\ |(\Gamma h)(z)| &\le \frac{2^{\alpha+1}}{\alpha}R^\alpha\|h\|_{C^{0,\alpha}(B_R(0))} \\ \frac{|(Ah)(z_2) - (Ah)(z_1)|}{|z_2 - z_1|^\alpha} &\le 2\|h\|_{C^0(B_R(0))} + \frac{2^{\alpha+2}}{\alpha}R^\alpha\|h\|_{C^{0,\alpha}(B_R(0))} \\ \frac{|(\Gamma h)(z_2) - (\Gamma h)(z_1)|}{|z_2 - z_1|^\alpha} &\le C_\alpha\,\|h\|_{C^{0,\alpha}(B_R(0))}.\end{aligned}$$

Recalling that

$$\|h\|_{C^{1,\alpha}(B_R(0))} := \|h\|_{C^0(B_R(0))} + \|\partial h\|_{C^{0,\alpha}(B_R(0))} + \|\bar\partial h\|_{C^{0,\alpha}(B_R(0))}$$

and

$$\|k\|_{C^{0,\alpha}(B_R(0))} := \|k\|_{C^0(B_R(0))} + \sup_{z_1 \ne z_2}\frac{|k(z_2) - k(z_1)|}{|z_2 - z_1|^\alpha}$$

and that the Hölder norm satisfies

$$\|hk\|_{C^{0,\alpha}(B_R(0))} \le C\,\|h\|_{C^{0,\alpha}(B_R(0))}\,\|k\|_{C^{0,\alpha}(B_R(0))}$$

for a suitable constant C depending only on α and R, the asserted inequality for the operator norm of $T_2 - T_1$ follows. In the same way, we obtain

$$\|g_2 - g_1\|_{C^{1,\alpha}(B_R(0))} \le c\,\|\delta_2 - \delta_1\|_{C^{0,\alpha}(B_R(0))}.$$

Since

$$\|w_2 - w_1\|_{C^{1,\alpha}(B_R(0))} \leq \|(T_2 - T_1)w_2\|_{C^{1,\alpha}(B_R(0))} + \theta \, \|w_2 - w_1\|_{C^{1,\alpha}(B_R(0))} + \|g_2 - g_1\|_{C^{1,\alpha}(B_R(0))}$$

and since $\theta < 1$, we obtain the assertion of the theorem for $k = 1$. Because derivatives of w again satisfy an equation of the form $\overline{\partial} w(z) = \mu(z)\partial w(z) + \gamma(z)w(z) + \delta(z)$, we can proceed by iteration. This is carried out in [11, Lecture 5]. □

3.2. A uniform L^2-bound for the harmonic forms and uniform convergence

PROPOSITION 3.8
Let $(S, j_0, \Gamma, \tilde{u}_0, \gamma_0)$ be a solution of (1.1) defined on $\dot{S}$ which is everywhere transverse to the Reeb vector field. Assume that $(S, j_f, \Gamma, \tilde{u} = (a, u), \gamma)$ is another smooth solution, where u is given by

$$u(z) = \phi_{f(z)}\big(u_0(z)\big)$$

for a suitable smooth bounded function $f : S \to \mathbb{R}$. Then we have

$$\|\gamma\|_{L^2, j_f} \leq \|u_0^*\lambda\|_{L^2, j_f}, \tag{3.7}$$

where

$$\|\sigma\|_{L^2, j_f} := \Big(\int_{\dot{S}} \sigma \circ j_f \wedge \sigma \Big)^{1/2}$$

(with σ a 1-form on $\dot{S}$).

Proof
Using the differential equation $u^*\lambda \circ j_f = da + \gamma$ and $u^*\lambda = u_0^*\lambda + df$, we compute

$$\int_{\dot{S}} u^*\lambda \wedge \gamma = \int_{\dot{S}} u_0^*\lambda \wedge \gamma + \int_{\dot{S}} d(f\gamma) = \int_{\dot{S}} u_0^*\lambda \wedge \gamma$$

and

$$\int_{\dot{S}} u^*\lambda \wedge \gamma = \int_{\dot{S}} da \wedge \gamma \circ j_f - \|\gamma\|^2_{L^2, j_f} = -\|\gamma\|^2_{L^2, j_f}.$$

The integral $\int_S d(f\gamma)$ vanishes by Stokes's theorem since $f\gamma$ is a smooth 1-form on the closed surface S. The form $da \wedge \gamma \circ j_f$ is not smooth on S, but the integral vanishes anyway for the following reason. As we have proved in the appendix, the form $\gamma \circ j_f$ is bounded near the punctures, and hence in local coordinates near a puncture it is of the form

$$\sigma = F(w_1, w_2)dw_1 + F_2(w_1, w_2)dw_2, \quad w_1 + iw_2 \in \mathbb{C},$$

where F_1, F_2 are smooth (except possibly at the origin) but bounded. Passing to polar coordinates via

$$\phi : [0, \infty) \times S^1 \longrightarrow \mathbb{C}\backslash\{0\}$$
$$\phi(s, t) = e^{-(s+it)} = w_1 + iw_2,$$

we see that $\phi^*\sigma$ has to decay at the rate e^{-s} for large s. The form da has $\gamma_1(r(s))\,ds$ as its leading term. Computing the integral $\int_\Gamma a(\gamma \circ j_f)$ over small loops Γ around the punctures and using Stokes's theorem, we conclude that the contribution from neighborhoods of the punctures can be made arbitrarily small. Therefore, the integral $\int_{\dot{S}} da \wedge \gamma \circ j_f$ must vanish.

If Ω is a volume form on S, then we may write $u_0^*\lambda \wedge \gamma = g \cdot \Omega$ for a suitable smooth function g. Defining

$$\int_{\dot{S}} |u_0^*\lambda \wedge \gamma| := \int_{\dot{S}} |g|\, \Omega,$$

we have

$$\begin{aligned} \|\gamma\|^2_{L^2, j_f} &= \left| \int_{\dot{S}} u_0^*\lambda \wedge \gamma \right| \\ &\leq \int_{\dot{S}} |u_0^*\lambda \wedge \gamma| \\ &\leq \|u_0^*\lambda\|_{L^2, j_f} \|\gamma\|_{L^2, j_f}, \end{aligned}$$

which implies the assertion. □

We resume the proof of the compactness result, Theorem 3.2. All the considerations which follow are local. The task is to improve the regularity of the limit f_{τ_0} and the nature of the convergence $f_\tau \to f_{\tau_0}$. Because the proof is somewhat lengthy, we organize it in several steps. For $\tau < \tau_0$, let now

$$\alpha_\tau : B \longrightarrow U_\tau \subset \mathbb{C}$$

be the conformal transformations as in Section 2, that is,

$$T\alpha_\tau(z) \circ i = j_\tau(\alpha_\tau(z)) \circ T\alpha_\tau(z), \quad z \in B.$$

The L^∞-bound (3.1) on the family of functions (f_τ) and the above L^2-bound imply convergence of the harmonic forms $\alpha_\tau^* \gamma_\tau$ after maybe passing to a subsequence.

PROPOSITION 3.9
Let τ_k' be a sequence converging to τ_0, and let $B' = B_{\varepsilon'}(0)$ with $\overline{B'} \subset B$. Then there is a subsequence $(\tau_k) \subset (\tau_k')$ such that the harmonic 1-forms $\alpha_{\tau_k}^* \gamma_{\tau_k}$ converge in $C^\infty(B')$.

Proof
First, the harmonic 1-forms $\alpha_\tau^* \gamma_\tau$ satisfy the same L^2-bound as in Proposition 3.8:

$$\begin{aligned}
\|\alpha_\tau^* \gamma_\tau\|_{L^2(B)}^2 &= \int_B \alpha_\tau^* \gamma_\tau \circ i \wedge \alpha_\tau^* \gamma_\tau \\
&= \int_B \alpha_\tau^* (\gamma_\tau \circ j_\tau) \wedge \alpha_\tau^* \gamma_\tau \\
&= \int_{U_\tau} \gamma_\tau \circ j_\tau \wedge \gamma_\tau \\
&\le \|u_0^* \lambda\|_{L^2, j_\tau} \\
&\le C,
\end{aligned}$$

where C is a constant depending only on λ and u_0 since

$$\sup_\tau \|j_\tau\|_{L^\infty(\dot{S})} < \infty.$$

We write

$$\alpha_\tau^* \gamma_\tau = h_\tau^1 \, ds + h_\tau^2 \, dt,$$

where h_τ^k, $k = 1, 2$ are harmonic and bounded in $L^2(B)$ independent of τ. If $y \in B$ and $B_R(y) \subset \overline{B_R(y)} \subset B$, then the classical mean-value theorem

$$h_\tau^k(y) = \frac{1}{\pi R^2} \int_{B_R(y)} h_\tau^k(x) dx$$

implies that, for any ball $B_\delta = B_\delta(y)$ with $B_\delta \subset \overline{B_\delta} \subset B$, we have the rather generous estimate

$$\|h_\tau^k\|_{C^0(B_\delta(y))} \le \frac{1}{\sqrt{\pi}\,\delta} \|h_\tau^k\|_{L^2(B)} \le \frac{\sqrt{C}}{\sqrt{\pi}\,\delta}.$$

With $y \in B$ and with $\nu = (\nu_1, \nu_2)$ being the unit outer normal to $\partial B_\delta(y)$, we get

$$\begin{aligned}\partial_s h^k_\tau(y) &= \frac{1}{\pi\delta^2}\int_{B_\delta(y)} \partial_s h^k_\tau(x)dx \\ &= \frac{1}{\pi\delta^2}\int_{B_\delta(y)} \operatorname{div}(h^k_\tau, 0)\,dx \\ &= \frac{1}{\pi\delta^2}\int_{\partial B_\delta(y)} h^k_\tau\,\nu_1\,ds\end{aligned}$$

and

$$\begin{aligned}|\nabla h^k_\tau(y)| &= \frac{1}{\pi\delta^2}\Big|\int_{\partial B_\delta(y)} h^k_\tau\,\nu\,ds\Big| \\ &\le \frac{2}{\delta}\|h^k_\tau\|_{C^0(\overline{B_\delta(y)})}\end{aligned}$$

so that, for $B' = B_{\varepsilon'}$, r being the radius of B, and $\delta = r - \varepsilon'$, we have

$$\|\nabla h^k_\tau\|_{C^0(\overline{B'})} \le \frac{2\sqrt{C}}{\sqrt{\pi}\,\delta^2}.$$

By iterating this procedure on nested balls we obtain τ-uniform $C^0(\overline{B'})$-bounds on all derivatives. Convergence as stated in the proposition then follows from the Ascoli-Arzela theorem. □

3.3. A uniform L^p-bound for the gradient

The first step of the regularity story is showing that the gradients of $a_\tau + if_\tau$ are uniformly bounded in $L^p(B')$ for some $p > 2$ and for any ball B' with $\overline{B'} \subset B$. It will soon become apparent why this gradient bound is necessary. Since we do not have a lot to start with, the proof will be indirect. Recall the differential equation (2.9)

$$\bar\partial_{j_\tau}(a_\tau + if_\tau) = u_0^*\lambda \circ j_\tau - i(u_0^*\lambda) - \gamma_\tau - i(\gamma_\tau \circ j_\tau),$$

where

$$\begin{aligned}j_\tau(z) = &\big(\pi_\lambda T u_0(z)\big)^{-1}\big(T\phi_{f_\tau(z)}(u_0(z))\big)^{-1} \\ &\times J\big(\phi_{f_\tau(z)}(u_0(z))\big)\,T\phi_{f_\tau(z)}\big(u_0(z)\big)\pi_\lambda T u_0(z).\end{aligned}$$

We set

$$\phi_\tau(z) := a_\tau(z) + i\,f_\tau(z), \quad z \in U_\tau$$

so that, for $z \in B$, we have

$$\begin{aligned}\overline{\partial}(\phi_\tau \circ \alpha_\tau)(z) &= \overline{\partial}_{j_\tau}\phi_\tau\big(\alpha_\tau(z)\big) \circ \partial_s\alpha_\tau(z) \\ &= \big(u_0^*\lambda \circ j_\tau - i(u_0^*\lambda)\big)_{\alpha_\tau(z)} \circ \partial_s\alpha_\tau(z) \\ &\quad - \Big((\alpha_\tau^*\gamma_\tau)(z) \cdot \frac{\partial}{\partial s} + i(\alpha_\tau^*\gamma_\tau)(z) \cdot \frac{\partial}{\partial t}\Big) \\ &=: \hat{F}_\tau(z) + \hat{G}_\tau(z) \\ &=: \hat{H}_\tau(z),\end{aligned} \tag{3.8}$$

and

$$\sup_\tau \|\hat{F}_\tau\|_{L^p(B)} < \infty \quad \text{for some } p > 2 \tag{3.9}$$

since $\alpha_\tau \to \alpha_{\tau_0}$ in $W^{1,p}(B)$ and $\sup_\tau \|j_\tau\|_{L^\infty} < \infty$. We also have

$$\sup_\tau \|\hat{G}_\tau\|_{C^k(B')} < \infty \tag{3.10}$$

for any ball $B' \subset \overline{B'} \subset B$ and any integer $k \geq 0$ in view of Proposition 3.9 (the proposition asserts uniform convergence after passing to a suitable subsequence, but uniform bounds on all derivatives are established in the proof). We claim now that, for every ball $B' \subset \overline{B'} \subset B$, there is a constant $C_{B'} > 0$ such that

$$\|\nabla(\phi_\tau \circ \alpha_\tau)\|_{L^p(B')} \leq C_{B'}, \quad \forall\, \tau \in [0, \tau_0). \tag{3.11}$$

Arguing indirectly, we may assume that there is a sequence $\tau_k \nearrow \tau_0$ such that

$$\|\nabla(\phi_{\tau_k} \circ \alpha_{\tau_k})\|_{L^p(B')} \to \infty \quad \text{for some ball } B' \subset \overline{B'} \subset B. \tag{3.12}$$

Now define

$$\varepsilon_k := \inf\Big\{\varepsilon > 0 \,\Big|\, \exists\, x \in B' : \|\nabla(\phi_{\tau_k} \circ \alpha_{\tau_k})\|_{L^p(B_\varepsilon(x))} \geq \varepsilon^{2/p-1}\Big\},$$

which are positive numbers since $\varepsilon^{(2/p)-1} \to +\infty$. Because we assumed (3.12) we must have $\inf_k \varepsilon_k = 0$, hence we will assume that $\varepsilon_k \to 0$. Otherwise, if $\varepsilon_0 = (1/2)\inf_k \varepsilon_k > 0$, then we cover $\overline{B'}$ with finitely many balls of radius ε_0, and we would get a k-uniform L^p-bound on each of them, contradicting (3.12). We claim that

$$\|\nabla(\phi_{\tau_k} \circ \alpha_{\tau_k})\|_{L^p(B_{\varepsilon_k}(x))} \leq \varepsilon_k^{(2/p)-1}, \quad \forall\, x \in B'. \tag{3.13}$$

Otherwise, we could find $y \in B'$ so that

$$\|\nabla(\phi_{\tau_k} \circ \alpha_{\tau_k})\|_{L^p(B_{\varepsilon_k}(y))} > \varepsilon_k^{(2/p)-1},$$

and we would still have the same inequality for a slightly smaller $\varepsilon_k' < \varepsilon_k$, contradicting the definition of ε_k. We now claim that there is a point $x_k \in B'$ with

$$\|\nabla(\phi_{\tau_k} \circ \alpha_{\tau_k})\|_{L^p(B_{\varepsilon_k}(x_k))} = \varepsilon_k^{(2/p)-1}. \tag{3.14}$$

Indeed, pick sequences $\delta_l \searrow \varepsilon_k$ and $y_l \in B'$ so that

$$\|\nabla(\phi_{\tau_k} \circ \alpha_{\tau_k})\|_{L^p(B_{\delta_l}(y_l))} \geq \delta_l^{(2/p)-1}.$$

We may assume that the sequence (y_l) converges. Denoting its limit by x_k, we obtain

$$\|\nabla(\phi_{\tau_k} \circ \alpha_{\tau_k})\|_{L^p(B_{\varepsilon_k}(x_k))} \geq \varepsilon_k^{(2/p)-1},$$

and (3.14) follows from (3.13). Hence there is a sequence $(x_k) \subset B'$ for which (3.14) holds. We may assume that the sequence $(x_k) \subset B'$ converges and (without loss of generality) also that $\lim_{k\to\infty} x_k = 0$. Let $R > 0$, and we define for $z \in B_R(0)$ the functions

$$\xi_k(z) := (\phi_{\tau_k} \circ \alpha_{\tau_k})\big(x_k + \varepsilon_k(z - x_k)\big),$$

which makes sense if k is sufficiently large. The transformation

$$\Phi : x \longmapsto x_k + \varepsilon_k(x - x_k)$$

satisfies $\Phi(B_1(x_k)) = B_{\varepsilon_k}(x_k)$ and $\Phi(B_1(y)) \subset B_{\varepsilon_k}(x_k + \varepsilon_k(y - x_k))$ so that

$$\begin{aligned}\int_{B_{\varepsilon_k}(x_k)} |\nabla(\phi_{\tau_k} \circ \alpha_{\tau_k})(x)|^p dx &= \varepsilon_k^2 \int_{B_1(x_k)} |\nabla(\phi_{\tau_k} \circ \alpha_{\tau_k})\big(x_k + \varepsilon_k(z - x_k)\big)|^p dz \\ &= \varepsilon_k^2 \int_{B_1(x_k)} \varepsilon_k^{-p} |\nabla \xi_k(z)|^p dz\end{aligned}$$

and

$$\begin{aligned}\|\nabla \xi_k\|_{L^p(B_1(x_k))} &= \varepsilon_k^{1-(2/p)} \|\nabla(\phi_{\tau_k} \circ \alpha_{\tau_k})\|_{L^p(B_{\varepsilon_k}(x_k))} \\ &= 1\end{aligned} \tag{3.15}$$

by (3.14) and, for any y for which $\xi_k|_{B_1(y)}$ is defined and for large enough k, we have

$$\|\nabla \xi_k\|_{L^p(B_1(y))} \leq \varepsilon_k^{1-(2/p)} \|\nabla(\phi_{\tau_k} \circ \alpha_{\tau_k})\|_{L^p(B_{\varepsilon_k}(x_k + \varepsilon_k(y - x_k)))} \leq 1 \tag{3.16}$$

by (3.13). The functions ξ_k satisfy the equation

$$\bar{\partial} \xi_k(z) = \varepsilon_k \hat{F}_{\tau_k}\big(x_k + \varepsilon_k(z - x_k)\big) + \varepsilon_k \hat{G}_{\tau_k}\big(x_k + \varepsilon_k(z - x_k)\big) =: H_{\tau_k}(z) \tag{3.17}$$

and, for every $R > 0$, we have

$$\sup_k \|\nabla\xi_k\|_{L^p(B_R(0))} < \infty, \qquad \|\nabla\xi_k\|_{L^p(B_2(0))} \geq 1 \tag{3.18}$$

because of (3.16) and (3.15) since $B_1(x_k) \subset B_2(0)$ for large k. The upper bound on $\|\nabla\xi_k\|_{L^p(B_R(0))}$ depends on how many balls $B_1(y)$ are needed to cover $B_R(0)$. We compute for $\rho > 0$

$$\|H_{\tau_k}\|_{L^p(B_\rho(x_k))} = \varepsilon_k^{1-(2/p)} \|\hat{H}_{\tau_k}\|_{L^p(B_{\rho\varepsilon_k}(x_k))},$$

with $\hat{H}_{\tau_k}$ as in (3.8). We conclude from $p > 2$, (3.9), and (3.10) that

$$\|H_{\tau_k}\|_{L^p(B_R(0))} \longrightarrow 0$$

for any $R > 0$ as $k \to \infty$. Defining

$$X^{l,p} := \{\psi \in W^{l,p}(B, \mathbb{C}^2) \mid \psi(0) = 0,\ \psi(\partial B) \subset \mathbb{R}^2\},\ l \geq 1,\ \ B \subset \mathbb{C} \text{ a ball},$$

the Cauchy-Riemann operator

$$\bar{\partial} : X^{l,p} \longrightarrow W^{l-1,p}(B, \mathbb{C}^2)$$

is a bounded bijective linear map. By the open mapping principle, we have the following estimate:

$$\|\psi\|_{l,p,B} \leq C\, \|\bar{\partial}\psi\|_{l-1,p,B}, \quad \forall\, \psi \in X^{l,p}. \tag{3.19}$$

Let $R' \in (0, R)$. Now pick a smooth function $\beta : \mathbb{R}^2 \to [0, 1]$ with $\beta|_{B_{R'}(0)} \equiv 1$ and $\text{supp}(\beta) \subset B_R(0)$. Define

$$\zeta_k(z) := \text{Re}\big(\xi_k(z) - \xi_k(0)\big) + i\beta(z)\,\text{Im}\big(\xi_k(z) - \xi_k(0)\big).$$

We note that

$$\sup_k \|\,\text{Im}(\xi_k)\|_{L^p(B_R(0))} \leq C_R$$

with a suitable constant $C_R > 0$ because of the uniform bound

$$\sup_{k,R} \|\,\text{Im}(\xi_k)\|_{L^\infty(B_R(0))} < \infty.$$

Using (3.19), we then obtain

$$\begin{aligned}\|\xi_k - \xi_k(0)\|_{1,p,B_{R'}(0)} &\le \|\zeta_k\|_{1,p,B_R(0)} \\ &\le C\,\|\overline{\partial}\zeta_k\|_{L^p(B_R(0))} \\ &\le C_{R'}(\|H_{\tau_k}\|_{L^p(B_R(0))} + \|\nabla\xi_k\|_{L^p(B_R(0))} \\ &\quad + \|\operatorname{Im}(\xi_k) - \operatorname{Im}(\xi_k)(0)\|_{L^p(B_R(0))})\end{aligned} \tag{3.20}$$

because of

$$\overline{\partial}\zeta_k = H_{\tau_k} + i(\beta - 1)\overline{\partial}\big(\operatorname{Im}(\xi_k)\big) + i\,\overline{\partial}\beta\,\big(\operatorname{Im}(\xi_k) - \operatorname{Im}(\xi_k)(0)\big).$$

Hence the sequence $(\xi_k - \xi_k(0))$ is uniformly bounded in $W^{1,p}(B_{R'}(0))$, and in particular, it has a subsequence which converges in $C^\alpha(\overline{B_{R'}(0)})$ for $0 < \alpha < 1 - \frac{2}{p}$ and also in $L^p(B_{R'}(0))$. For $R'' \in (0, R')$, we now use the regularity estimate

$$\begin{aligned}\|\xi_l - \xi_k - (\xi_l - \xi_k)(0)\|_{1,p,B_{R''}(0)} \le{}& c\,\|H_{\tau_l} - H_{\tau_k}\|_{L^p(B_{R'}(0))} \\ &+ c\,\|\xi_l - \xi_k - (\xi_l - \xi_k)(0)\|_{L^p(B_{R'}(0))},\end{aligned} \tag{3.21}$$

where $c = c(p, R', R'') > 0$. This follows from (3.19) applied to $\psi = \beta(\xi_l - \xi_k - (\xi_l - \xi_k)(0))$, where β is a smooth cutoff function with support in $B_{R'}$ and where $\beta \equiv 1$ on $B_{R''}$. We may then assume that the right-hand side of (3.21) converges to zero as $k, l \to \infty$. This argument can be carried out for any triple $0 < R'' < R' < R$. Hence the sequence $\xi_k - \xi_k(0)$ converges in $W^{1,p}_{\text{loc}}(\mathbb{C})$ to some limit $\xi : \mathbb{C} \to \mathbb{C}$ which solves $\bar{\partial}\xi = 0$ in the sense of distributions. Therefore, it is an entire holomorphic function. Because the imaginary parts of ξ_k are uniformly bounded, this also applies to $\operatorname{Im}(\xi)$. Liouville's theorem for harmonic functions then implies that $\operatorname{Im}(\xi)$ must be constant, hence ξ is constant as well. On the other hand, ξ cannot be constant since it satisfies $\|\nabla\xi\|_{L^p(B_2(0))} \ge 1$. This contradiction finally disproves our assertion (3.12). We summarize with the following.

PROPOSITION 3.10
For every ball B' with $\overline{B'} \subset B$ we have

$$\sup_\tau \|\nabla(\phi_\tau \circ \alpha_\tau)\|_{L^p(B')} < \infty.$$

Remark 3.11
After establishing the estimates (3.18) for $\nabla\xi_k$, we could have derived a k uniform $W^{1,p}$-bound for ξ_k minus its average $\overline{\xi_k}$ over the ball $B_R(0)$ via Poincaré's inequality. We could have derived $W^{1,p}(B_R(0))$ convergence of $\xi_k - \overline{\xi_k}$, but not convergence in

$W^{1,p}_{\mathrm{loc}}(\mathbb{C})$ since the sequence $(\xi_k - \overline{\xi_k})$ depends on the choice of the ball $B_R(0)$. Our sequence $\xi_k - \xi_k(0)$ has a convergent subsequence on any ball.

3.4. Convergence in $W^{1,p}(B')$

Pick a sequence $\tau_k \nearrow \tau_0$. We claim that the sequence $(\hat{F}_{\tau_k})$ converges in $L^p(B)$ maybe after passing to a suitable subsequence (recall that, so far, we only have the uniform bound (3.9)). The functions $\hat{F}_{\tau_k}$ converge pointwise almost everywhere after passing to some subsequence. Indeed, the sequence $\{(u_0^*\lambda \circ j_{\tau_k} - i(u_0^*\lambda))_{\alpha_{\tau_k}(z)}\}$ converges already pointwise since j_{τ_k} and α_{τ_k} do (recall that the sequence (α_{τ_k}) converges in $W^{1,p}(B)$ and therefore uniformly). The sequence $(\partial_s \alpha_{\tau_k})$ converges in $L^p(B)$ and therefore pointwise almost everywhere after passing to a suitable subsequence. Then by Egorov's theorem, for any $\delta > 0$, there is a subset $E_\delta \subset B$ with $|B \backslash E_\delta| \le \delta$ so that the sequence $\hat{F}_{\tau_k}$ converges uniformly on E_δ. Let α be the L^p-limit of the sequence $(\partial_s \alpha_{\tau_k})$, and let $\varepsilon > 0$. We introduce

$$C := 2 \sup_{0 \le \tau < \tau_0} \left\| (u_0^*\lambda \circ j_\tau - i(u_0^*\lambda))_{\alpha_\tau(z)} \right\|_{L^\infty(B)}.$$

Now pick $\delta > 0$ sufficiently small such that

$$\|\alpha\|_{L^p(B\backslash E_\delta)} \le \frac{\varepsilon}{3\,C}.$$

Now choose $k_0 \ge 0$ so large that, for all $k \ge k_0$, we have

$$\|\partial_s \alpha_{\tau_k} - \alpha\|_{L^p(B)} \le \frac{\varepsilon}{3\,C} \qquad \text{and} \qquad \|\hat{F}_{\tau_k} - \hat{F}_{\tau_l}\|_{L^\infty(E_\delta)} \le \frac{\varepsilon}{3\,|B|}.$$

Then, if $k, l \ge k_0$, we have

$$\begin{aligned} \|\hat{F}_{\tau_k} - \hat{F}_{\tau_l}\|_{L^p(B)} &\le \|\hat{F}_{\tau_k} - \hat{F}_{\tau_l}\|_{L^p(E_\delta)} + \|\hat{F}_{\tau_k} - \hat{F}_{\tau_l}\|_{L^p(B\backslash E_\delta)} \\ &\le |E_\delta|\, \|\hat{F}_{\tau_k} - \hat{F}_{\tau_l}\|_{L^\infty(E_\delta)} + 2 \sup_{k \ge k_0} \|\hat{F}_{\tau_k}\|_{L^p(B\backslash E_\delta)} \\ &\le |B|\, \|\hat{F}_{\tau_k} - \hat{F}_{\tau_l}\|^p_{L^\infty(E_\delta)} + C \cdot \sup_{k \ge k_0} \|\partial_s \alpha_{\tau_k}\|_{L^p(B\backslash E_\delta)} \\ &\le \varepsilon \end{aligned}$$

proving the claim.

Recalling that $\phi_\tau = a_\tau + i f_\tau$ and that the family f_τ satisfies a uniform L^∞-bound, we have

$$\sup_\tau \|\operatorname{Im}(\phi_\tau \circ \alpha_\tau)\|_{L^\infty(B)} < \infty.$$

Now pick three balls $B''' \subset B'' \subset B' \subset B$ such that the closure of one is contained in the next. Our aim is to establish $W^{1,p}(B''')$-convergence of a subsequence of the sequence $(\phi_{\tau_k} \circ \alpha_{\tau_k})$. By Proposition 3.10, we have a uniform $L^p(B')$-bound on the gradient. If $\beta : \mathbb{R}^2 \to [0,1]$ is a smooth function with $\operatorname{supp}(\beta) \subset B'$ and $\beta|_{B''} \equiv 1$ and if

$$\zeta_\tau = \operatorname{Re}\big(\phi_\tau \circ \alpha_\tau - \phi_\tau(0)\big) + i\beta \operatorname{Im}\big(\phi_\tau \circ \alpha_\tau - \phi_\tau(0)\big),$$

then we proceed in the same way as in (3.20), and we obtain

$$\|\varphi_k\|_{1,p,B''} \le C\,(\|\hat{H}_{\tau_k}\|_{L^p(B')} + \|\nabla(\phi_{\tau_k} \circ \alpha_{\tau_k})\|_{L^p(B')} + \|\operatorname{Im}(\varphi_k)\|_{L^p(B')}),$$

where we wrote

$$\varphi_k := \phi_{\tau_k} \circ \alpha_{\tau_k} - (\phi_{\tau_k} \circ \alpha_{\tau_k})(0),$$

and where $C > 0$ is a constant depending only on p, B', and B''. The sequence (φ_k) is then uniformly bounded in $W^{1,p}(B'')$ by Proposition 3.10, and it converges in $L^p(B'')$ after passing to a suitable subsequence. We now use the regularity estimate

$$\begin{aligned}\|\varphi_l - \varphi_k\|_{1,p,B'''} \le C\,(&\|\hat{F}_{\tau_l} - \hat{F}_{\tau_k}\|_{L^p(B'')} \\ &+ \|\hat{G}_{\tau_l} - \hat{G}_{\tau_k}\|_{L^p(B'')} + \|\varphi_l - \varphi_k\|_{L^p(B'')}).\end{aligned} \tag{3.22}$$

Since the right-hand side converges to zero as $k, l \to \infty$, we obtain the following.

PROPOSITION 3.12
For every ball $B' \subset \overline{B'} \subset B$, the sequence $(\phi_{\tau_k} \circ \alpha_{\tau_k} - \phi_{\tau_k}(0))$ has a subsequence which converges in $W^{1,p}(B')$.

3.5. Improving regularity using both the Beltrami and Cauchy-Riemann equations

In order to improve the convergence of the conformal transformations α_{τ_k}, we need to improve the convergence of the maps $\mu_{\tau_k} \to \mu_{\tau_0}$ and the regularity of its limit. It is known that the inverses $\alpha_{\tau_k}^{-1}$ of the conformal transformations α_{τ_k} also satisfy a Beltrami equation (see [9])

$$\bar{\partial}\alpha_\tau^{-1} = \nu_\tau\, \partial\alpha_\tau^{-1},$$

where

$$\nu_\tau(z) = -\frac{\partial\alpha_\tau(\alpha_\tau^{-1}(z))}{\overline{\partial\overline{\alpha}_\tau(\alpha_\tau^{-1}(z))}}\mu_\tau\big(\alpha_\tau^{-1}(z)\big)$$

(follows from $0 = \overline{\partial}(\alpha_\tau^{-1} \circ \alpha_\tau) = \overline{\partial}\alpha_\tau^{-1}(\alpha_\tau)\overline{\partial}\overline{\alpha}_\tau + \partial\alpha_\tau^{-1}(\alpha_\tau)\overline{\partial}\alpha_\tau$). After passing to a suitable subsequence, we may assume that $\partial\alpha_{\tau_k}$ and $\overline{\partial}\alpha_{\tau_k}$ converge pointwise almost everywhere since they converge in $L^p(B)$. Hence we may assume that the sequence (ν_{τ_k}) also converges pointwise almost everywhere. We also have

$$|\nu_\tau(z)| \leq \left|\mu_\tau\big(\alpha^{-1}(z)\big)\right|,$$

and hence ν_τ satisfies the same L^∞-bound as μ_τ. By Lemma 3.6, we conclude that

$$\alpha_{\tau_k}^{-1} \longrightarrow \alpha_1^{-1} \text{ in } W^{1,p}(B)$$

with the same $p > 2$ as in Lemma 3.6 applied to the functions μ_τ. After passing to some subsequence, the sequence

$$(\varphi_k) := \big(\phi_{\tau_k} - a_{\tau_k}(0)\big) \circ \alpha_{\tau_k}$$

converges in $W^{1,p}(B')$ for any ball $\overline{B'} \subset B$ by Proposition 3.12. Indeed, the expression $\phi_{\tau_k} \circ \alpha_{\tau_k} - \phi_{\tau_k}(0)$ and φ_k differ by a constant term $if_{\tau_k}(0)$, but the sequence $(if_{\tau_k}(0))$ has a convergent subsequence.

We would like to derive a decent notion of convergence for the sequence $(\varphi_{\tau_k} \circ \alpha_{\tau_k}^{-1})$, but the space $W^{1,p}$ is not well behaved under compositions. The composition of two functions of class $W^{1,p}$ is only in $W^{1,p/2}$. Since we cannot choose $p > 2$ freely, we rather carry out the argument in Hölder spaces. By the Sobolev embedding theorem and Rellich compactness, we may assume that the sequences (φ_k) and $(\alpha_{\tau_k}^{-1})$ converge in $C^{0,\alpha}(B')$ for any ball $B' \subset \overline{B'} \subset B$ and $0 < \alpha \leq 1 - (2/p)$. We conclude from the inequality

$$\|f \circ g\|_{C^{0,\gamma\delta}(B')} \leq \|f\|_{C^{0,\gamma}(B')}\|g\|_{C^{0,\delta}(B')}, \quad \forall\, f \in C^{0,\gamma}(B'),\ g \in C^{0,\delta}(B'),$$

where $0 < \gamma, \delta \leq 1$, that the sequence $(\phi_{\tau_k} - a_{\tau_k}(0))$ converges in $C^{0,\alpha^2}(B')$. In particular, any sequence (f_{τ_k}), $\tau_k \nearrow \tau_0$ now converges in the C^{0,α^2}-norm to f_{τ_0}. Hölder spaces are well behaved with respect to multiplication, that is,

$$\|fg\|_{C^{0,\gamma}(B')} \leq 2\,\|f\|_{C^{0,\gamma}(B')}\|g\|_{C^{0,\gamma}(B')},$$

and composition with a fixed smooth function maps $C^{0,\gamma}(B')$ into itself. It then follows from the definition of the complex structure j_τ and from the definition of μ_τ that $\mu_{\tau_k} \to \mu_{\tau_0}$ in the C^{0,α^2}-norm as well. We conclude from Theorem 3.7, the classical regularity result for the Beltrami equation, that $\alpha_{\tau_k} \to \alpha_{\tau_0}$ in the C^{1,α^2}-norm. The regularity estimate for the Cauchy-Riemann operator (3.22) is also valid in Hölder spaces, that is,

$$\|\varphi_l - \varphi_k\|_{C^{k+1,\gamma}(B''')} \le C\,(\|\hat{F}_{\tau_l} - \hat{F}_{\tau_k}\|_{C^{k,\gamma}(B'')} + \|\hat{G}_{\tau_l} - \hat{G}_{\tau_k}\|_{C^{k,\gamma}(B'')} + \|\varphi_l - \varphi_k\|_{C^{k,\gamma}(B'')}).$$

The sequence $(\hat{F}_{\tau_k})$ now converges in the C^{0,α^2}-norm, and the sequence $(\hat{G}_{\tau_k})$ converges in any Hölder norm. We obtain with the above regularity estimate C^{1,α^2}-convergence of the sequence (φ_k), and composing with $\alpha_{\tau_k}^{-1}$ yields C^{1,α^4}-convergence of (f_{τ_k}) and (μ_{τ_l}). Invoking Theorem 3.7 again then improves the convergence of the transformations α_{τ_k}, $\alpha_{\tau_k}^{-1}$ to C^{2,α^4}. We now iterate the procedure using the regularity estimate for the Cauchy-Riemann operator in Hölder space and the estimate for the Beltrami equation in Theorem 3.7.

Theorem 3.2 follows if we apply the implicit function theorem to the limit solution $(S, j_{\tau_0}, \tilde{u}_{\tau_0} = (a_{\tau_0}, u_{\tau_0}), \gamma_{\tau_0})$, hence we obtain the same limit for every sequence $\{\tau_k\}$, and we obtain convergence in C^∞.

4. Conclusion

The following remarks tie together the loose ends and prove the main result, Theorem 1.6. We start with a closed 3-dimensional manifold with contact form λ'. Giroux's theorem, Theorem 1.4, then permits us to change the contact form λ' to another contact form λ such that $\ker\lambda = \ker\lambda'$ and such that there is a supporting open book decomposition with binding K consisting of periodic orbits of the Reeb vector field of λ. Invoking Proposition 2.4, we construct a family of 1-forms $(\lambda_\delta)_{0\le\delta<1}$ which are contact forms except λ_0, and the above open book supports $\ker\lambda_\delta$ as well if $\delta \ne 0$. By the uniqueness part of Giroux's theorem, $(M, \ker\lambda)$ and $(M, \ker\lambda_\delta)$ are diffeomorphic for $\delta \ne 0$, hence we may assume without loss of generality that $\lambda = \lambda_\delta$. Proposition 2.6 then permits us to turn the Giroux leaves into holomorphic curves for data associated with the confoliation form λ_0. Picking one Giroux leaf, the implicit function theorem, Theorem 2.8, then allows us to deform it into solutions to our PDE (1.1) for small $\delta \ne 0$. Leaving such a parameter δ fixed from now, and denoting the corresponding solution by $(\tilde{u}_0, \gamma_0, j_0)$, Theorem 2.8 then delivers more solutions $(\tilde{u}_\tau, \gamma_\tau, j_\tau)_{0\le\tau<\tau_0}$. The leaves $u_\tau(\dot{S})$ are all global surfaces of section, recalling that they are of the form $u_\tau = \phi_{f_\tau}(u_0)$. Theorem 2.8 also implies that $f_\tau < f_{\tau'}$ if $\tau < \tau'$. The compactness result, Theorem 3.2, then implies that there is a last solution for $\tau = \tau_0$ as well, and that either $u_{\tau_0}(\dot{S})$ is disjoint from $u_0(\dot{S})$ or agrees with it. In the latter case, the proof of Theorem 1.6 is complete. In the first case, we apply Theorem 2.8 again to $(\tilde{u}_{\tau_0}, \gamma_{\tau_0}, j_{\tau_0})$, producing a larger family of solutions. Because $\tau \mapsto f_\tau(z)$ is strictly monotone for each $z \in S$ and because the return time for each point on $u_0(\dot{S})$

is finite, the images of u_τ and u_0 must agree for some sufficiently large τ, concluding the proof. □

Appendix. Some local computations near the punctures

In this appendix, we present some local computations needed for the proof of Theorem 2.8. The issue is to show that the 1-forms

$$u_0^*\lambda \circ j_f - da_0 \qquad \text{and} \qquad u_0^*\lambda + da_0 \circ j_f$$

are bounded on $\dot{S}$. We obtain in the second case of the theorem

$$\begin{aligned} u_0^*\lambda \circ j_f - da_0 &= u_0^*\lambda \circ (j_f - j_g) + \gamma_0 \\ &= dg \circ (j_f - j_g) + \gamma_0 + v_0^*\lambda \circ (j_f - j_g). \end{aligned}$$

The first case can be treated as a special case: here the objective is to show that the 1-form $v_0^*\lambda \circ (j_f - i) = v_0^*\lambda \circ (j_f - j_0)$ is bounded near the punctures. We again drop the subscript δ in the notation since we are only concerned with a local analysis near the binding, and all the forms λ_δ are identical there. We use coordinates (θ, r, ϕ) near the binding. The contact structure is then generated by

$$\eta_1 = \frac{\partial}{\partial r} = (0, 1, 0), \qquad \eta_2 = -\gamma_2 \frac{\partial}{\partial \theta} + \gamma_1 \frac{\partial}{\partial \phi} = (-\gamma_2, 0, \gamma_1).$$

The projection onto the contact planes along the Reeb vector field is then given by

$$\pi_\lambda(v_1, v_2, v_3) = \frac{1}{\mu}(v_1\gamma_1' + v_3\gamma_2')\,\eta_2 + v_2\,\eta_1, \quad \text{with } \mu = \gamma_1\gamma_2' - \gamma_1'\gamma_2,$$

and the flow of the Reeb vector field is given by

$$\phi_t(\theta, r, \phi) \;=\; \big(\theta + \alpha(r)t, r, \phi + \beta(r)t\big),$$

where

$$\alpha(r) \;=\; \frac{\gamma_2'(r)}{\mu(r)} \qquad \text{and} \qquad \beta(r) \;=\; -\frac{\gamma_1'(r)}{\mu(r)}.$$

The linearization of the flow $T\phi_\tau(\theta, r, \phi)$ preserves the contact structure. In the basis $\{\eta_1, \eta_2\}$ it is given by

$$T\phi_\tau(\theta, r, \phi) \;=\; \begin{pmatrix} 1 & 0 \\ \tau A(r) & 1 \end{pmatrix} \quad \text{with } A(r) = \frac{1}{\mu^2(r)}\big(\gamma_2''(r)\gamma_1'(r) - \gamma_1''(r)\gamma_2'(r)\big).$$

The complex structure(s) we chose earlier in (2.7) had the following form near the binding with respect to the basis $\{\eta_1, \eta_2\}$:

$$J(\theta, r, \phi) = \begin{pmatrix} 0 & -r\gamma_1(r) \\ \frac{1}{r\gamma_1(r)} & 0 \end{pmatrix}.$$

The induced complex structure j_τ on the surface is then given by

$$j_\tau(z) = [\pi_\lambda T v_0(z)]^{-1}[T\phi_\tau\big(v_0(z)\big)]^{-1} J\big(\phi_\tau(v_0(z))\big) T\phi_\tau\big(v_0(z)\big)\pi_\lambda T v_0(z).$$

With $v_0(s,t) = (t, r(s), \alpha)$, we find that

$$\pi_\lambda T v_0(s,t) = \begin{pmatrix} r'(s) & 0 \\ 0 & \frac{\gamma_1'(s)}{\mu(r(s))} \end{pmatrix}$$

so that

$$\begin{aligned} j_\tau &= \begin{pmatrix} -\tau A(r) r\gamma_1(r) & -\frac{r\gamma_1(r)\gamma_1'(r)}{r'\mu(r)} \\ \frac{r'\mu(r)}{r\gamma_1(r)\gamma_1'(r)}(1+\tau^2 A^2(r) r^2\gamma_1^2(r)) & \tau A(r) r\gamma_1(r) \end{pmatrix} \\ &= \begin{pmatrix} -\tau A(r) r\gamma_1(r) & -1 \\ 1+\tau^2 A^2(r) r^2 \gamma_1^2(r) & \tau A(r) r\gamma_1(r) \end{pmatrix} \\ &= j_0 + \tau\, A(r)\,\gamma_1(r) \begin{pmatrix} -1 & 0 \\ \tau A(r)\gamma_1(r) & 1 \end{pmatrix} \end{aligned}$$

and

$$j_\tau - j_\sigma \;=\; A(r) r\gamma_1(r)(\tau-\sigma) \begin{pmatrix} -1 & 0 \\ (\tau+\sigma)A(r) r\gamma_1(r) & 1 \end{pmatrix},$$

using the fact that $r(s)$ satisfies the differential equation

$$r'(s) = \frac{\gamma_1'(r(s))\gamma_1(r(s)) r(s)}{\mu(r(s))}.$$

With $v_0^*\lambda = \gamma_1(r)\,dt$, we obtain

$$\begin{aligned} v_0^*\lambda \circ (j_\tau - j_\sigma)|_{(s,t)} = (\tau-\sigma) A\big(r(s)\big) r(s)\gamma_1^2\big(r(s)\big) \\ \cdot\big[(\tau+\sigma)A\big(r(s)\big) r(s)\gamma_1\big(r(s)\big) ds + dt\big]. \end{aligned}$$

Converting from coordinates (s,t) on the half-cylinder to Cartesian coordinates $x + iy = e^{-(s+it)}$ in the complex plane, we get, with $\rho = \sqrt{x^2+y^2}$,

$$ds = -\frac{1}{\rho^2}(x\,dx + y\,dy) \qquad \text{and} \qquad dt = -\frac{1}{\rho^2}(x\,dy - y\,dx).$$

Recall that $r(s) = c(s)e^{\kappa s}$, where $c(s)$ is a smooth function which converges to a constant as $s \to +\infty$, and we assumed that $\kappa \le -1/2$ and that $\kappa \notin \mathbb{Z}$. Another assumption was that $A(r) = O(r)$. Hence $r(s)$ is close to $\rho^{-\kappa}$ if s is large (and ρ is small) and $A(r(s)) = O(\rho^{-\kappa})$. Also recall that $\gamma_1(r(s)) = O(1)$. Summarizing, we need the expression

$$A\bigl(r(s)\bigr)r(s)\rho^{-2}\rho = O(\rho^{-2\kappa-1})$$

to be bounded, which amounts to $\kappa \le -1/2$. The same argument applied to the form $dg \circ (j_\tau - j_\sigma)$ leads to the same conclusion.

Acknowledgments. I am very grateful to Richard Siefring and Chris Wendl for explaining some of their work to me. Their results are indispensable for the arguments in this article. I would also like to thank Samuel Lisi for the numerous discussions we had about the subject of this article.

References

[1] C. ABBAS, *Pseudoholomorphic strips in symplectizations, II: Fredholm theory and transversality*, Comm. Pure Appl. Math. **57** (2004), 1–58. MR 2007355 58

[2] ———, *Pseudoholomorphic strips in symplectizations, III: Embedding properties and compactness*, J. Symplectic Geom. **2** (2004), 219–260. MR 2108375

[3] ———, *Introduction to compactness results in symplectic field theory*, in preparation. 32

[4] C. ABBAS, K. CIELIEBAK, and H. HOFER, *The Weinstein conjecture for planar contact structures in dimension three*, Comment. Math. Helv. **80** (2005), 771–793. MR 2182700 29, 31, 36

[5] C. ABBAS and H. HOFER, *Holomorphic curves and global questions in contact geometry*, in preparation. 56

[6] C. ABBAS, H. HOFER, and S. LISI, *Renormalization and energy quantization in Reeb dynamics*, in preparation. 29, 36

[7] ———, *Some applications of a homological perturbed Cauchy-Riemann equation*, in preparation. 29, 36

[8] L. V. AHLFORS, *Lectures on Quasiconformal Mappings*, Van Nostrand Math. Stud. **10**, D. Van Nostrand, Toronto, 1966. 59, 60

[9] L. V. AHLFORS and L. BERS, *Riemann's mapping theorem for variable metrics*, Ann. of Math. (2) **72** (1960), 385–404. MR 0115006 59, 60, 61, 62, 75

[10] J. W. ALEXANDER, *A lemma on systems of knotted curves*, Proc. Natl. Acad. Sci. USA **9** (1923), 93 – 95. 30

[11] L. BERS, *Riemann Surfaces*, lectures, New York University, 1957 – 1958.

[12] L. BERS and L. NIRENBERG, "On a representation theorem for linear elliptic systems with discontinuous coefficients and its applications" in *Convegno internazionale sulle equazioni lineari alle derivate parziali, Trieste (1954)*, 111 – 140, Cremonese, Rome, 1955. MR 0076981 60, 63, 64, 65, 66 60

[13] S.-S. CHERN, *An elementary proof of the existence of isothermal parameters on a surface*, Proc. Amer. Math. Soc. **6** (1955), 771 – 782. MR 0074856 60, 63, 64, 65

[14] J. B. ETNYRE, *Planar open book decompositions and contact structures*, Int. Math. Res. Not. IMRN **2004**, no. 79, 4255 – 4267. MR 2126827 31

[15] O. FORSTER, *Lectures on Riemann Surfaces*, Grad. Texts in Math. **81**, Springer, New York, 1981. MR 0648106 51

[16] E. GIROUX, "Géométrie de contact: De la dimension trois vers les dimensions supérieures" in *Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002)*, Higher Ed. Press, Beijing, 2002, 405 – 414. MR 1957051 30, 31

[17] H. HOFER, *Pseudoholomorphic curves in symplectizations with applications to the Weinstein conjecture in dimension three*, Invent. Math. **114** (1993), 515 – 563. MR 1244912 32

[18] ———, "Holomorphic curves and real three-dimensional dynamics" in *GAFA 2000 (Tel Aviv, 1999)*, Geom. Funct. Anal. **2000**, Special Volume, Part II, 674 – 704. MR 1826268

[19] H. HOFER, K. WYSOCKI, and E. ZEHNDER, *Properties of pseudoholomorphic curves in symplectizations, II: Embedding controls and algebraic invariants*, Geom. Funct. Anal. **5** (1995), 270 – 328. MR 1334869

[20] ———, *The dynamics on three-dimensional strictly convex energy surfaces*, Ann. of Math. (2) **148** (1998), 197 – 289. MR 1652928 35

[21] ———, *Properties of pseudoholomorphic curves in symplectizations,. III: Fredholm theory*, Progr. Nonlinear Differential Equations Appl. **35**, Birkhäuser, Basel, 1999, 381 – 475. MR 1725579 58

[22] ———, *Finite energy foliations of tight three-spheres and Hamiltonian dynamics*, Ann. of Math. (2) **157** (2003), 125 – 255. MR 1954266 35

[23] H. HOFER and E. ZEHNDER, *Symplectic Invariants and Hamiltonian Dynamics*, Birkhäuser Advanced Texts: Basler Lehrbücher, Birkhäuser, Basel, 1994. MR 1306732 52

[24] C. HUMMEL, *Gromov's Compactness Theorem for Pseudoholomorphic Curves*, Prog. Math. **151**, Birkhäuser, Basel, 1997. MR 1451624

[25] P. D. LAX, *Functional Analysis*, John Wiley, New York, 2001. MR 1892228 61

[26] L. LICHTENSTEIN, *Zur Theorie der konformen Abbildung nichtanalytischer, singularitätenfreier Flächenstücke auf ebene Gebiete*, J. Krak. Auz. (1916), 192 – 217. 60

[27] D. MCDUFF and D. SALAMON, *J-holomorphic Curves and Symplectic Topology*, Amer. Math. Soc. Colloq. Publ. **52**, Amer. Math. Soc., Providence, 2004. MR 2045629 56

[28] M. J. MICALLEF and B. WHITE, *The structure of branch points in minimal surfaces and in pseudoholomorphic curves*, Ann. of Math. (2) **141** (1995), 35 – 85. MR 1314031 56

[29] C. B. MORREY, *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc. **43**, no. 1 (1938), 126 – 166. MR 1501936 60

[30] D. ROLFSEN, *Knots and Links*, Mathematics Lecture Series **7**, Publish or Perish, Berkeley, Cali., 1976. MR 0515288 30

[31] R. SIEFRING, *Relative asymptotic behavior of pseudoholomorphic half-cylinders*, Comm. Pure Appl. Math. **61** (2008), 1631 – 1684. MR 2456182 52, 53, 58

[32] E. M. STEIN, *Singular Integrals and Differentiability Properties of Functions*, Princeton Math. Ser. **30**, Princeton Univ. Press, Princeton, 1970. MR 0290095 61

[33] C. H. TAUBES, *The Seiberg-Witten equations and the Weinstein conjecture*, Geom. Topol. **11** (2007), 2117 – 2202. MR 2350473 29, 36

[34] ———, *The Seiberg-Witten equations and the Weinstein conjecture, II: More closed integral curves of the Reeb vector field*, Geom. Topol. **13** (2009), 1337 – 1417. MR 2496048 29, 36

[35] W. P. THURSTON and H. E. WINKELNKEMPER, *On the existence of contact forms*, Proc. Amer. Math. Soc. **52** (1975), 345 – 347. MR 0375366 40

[36] A. J. TROMBA, *Teichmüller Theory in Riemannian Geometry*, Lectures Math. ETH Zürich, Birkhäuser, Basel, 1992. MR 1164870

[37] A. WEINSTEIN, *On the hypotheses of Rabinowitz' periodic orbit theorems*, J. Differential Equations **33** (1979), 353 – 358. MR 0543704

[38] C. WENDL, *Finite energy foliations on overtwisted contact manifolds*, Geom. Topol. **12** (2008) 531 – 616. MR 2390353 34, 35, 36

[39] ———, *Open book decompositions and stable Hamiltonian structures*, Expo. Math. **28** (2010), 187 – 197. MR 2671115 34, 36, 40

[40] ———, *Strongly fillable contact manifolds and J-holomorphic foliations*, Duke Math. J. **151** (2010), 337 – 387. MR 2605865 35

Department of Mathematics, Michigan State University, East Lansing, Michigan 48824, USA;
abbas@math.msu.edu